Optimizing an oriented convex hull with two directions

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Abstract

Given a set P of n points in the plane in general position, we generalize the rectilinear convex hull of P, $\mathcal{RH}(P)$, to the $\mathcal{O}^2_\beta$-convex hull of P, denoted by $\mathcal{O}^2_\beta\mathcal{H}(P)$, where the directions of two oriented lines, used as coordinate axes, form an angle $\beta \in [0, \pi]$. We show: (i) How this hull can be computed and maintained while β changes in $[0, \pi]$, and (ii) How to determine the angle β for which $\mathcal{O}^2_\beta\mathcal{H}(P)$ maximizes its area or minimizes its perimeter. Our algorithms run in optimal $\Theta(n \log n)$ time and $O(n)$ space.

1 Introduction

All the point sets P considered in this paper will be assumed in general position and such that no two elements of P lie on a horizontal line. Let $\mathcal{O}^k$ be a set of k lines in the plane through a common point. A region R in the plane is called *$\mathcal{O}^k$-convex* if its intersection with any line parallel to one in $\mathcal{O}^k$ is either empty or connected, see [4, 7].

Ottmann et al. [5] consider $k = 2$ with horizontal and vertical lines, showing how to compute the so-called rectilinear convex hull of P, denoted by $\mathcal{RH}(P)$, in optimal $\Theta(n \log n)$ time and $O(n)$ space. Rotating the set of two lines makes $\mathcal{RH}(P)$ change and the rotation for which $\mathcal{RH}(P)$ has minimum area was obtained in [1], in optimal $\Theta(n \log n)$ time and $O(n)$ space. See [2] for a generalization.

Here we also consider the case $k = 2$, with a set $\mathcal{O}^2$ composed of a horizontal line (oriented from left to right) and a second line (oriented from bottom to top) forming an angle β with the horizontal, see Figure 1 (left). Hence, we may denote $\mathcal{O}^2$ as $\mathcal{O}^2_\beta$.

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Following Ottmann [5], we define the $\mathcal{O}^2_\beta$-convex hull of a point set P as the intersection of all the connected supersets of P which are $\mathcal{O}^2_\beta$-convex, see Figure 1 (right). The $\mathcal{O}^2_\beta$-convex hull of a point set P will be denoted as $\mathcal{O}^2_\beta\mathcal{H}(P)$. In this paper we show algorithms for: (i) Computing and maintaining $\mathcal{O}^2_\beta\mathcal{H}(P)$ while β changes in $[0, \pi]$, and (ii) finding an angle $\beta \in [0, \pi]$ such that the area of $\mathcal{O}^2_\beta\mathcal{H}(P)$ is maximized or the (non-zero) perimeter of $\mathcal{O}^2_\beta\mathcal{H}(P)$ is minimized. Our algorithms run in $\Theta(n \log n)$ time and $O(n)$ space.

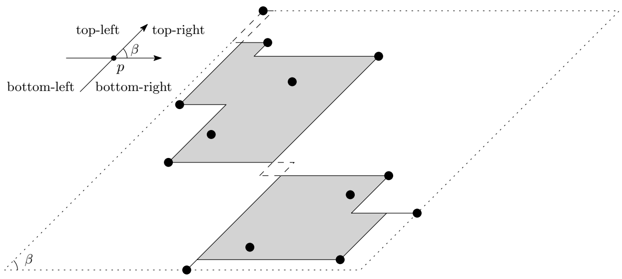

Figure 1: left: Example of $\mathcal{O}^2$. right: Example of $\mathcal{O}^2_\beta\mathcal{H}(P)$.

Let $\mathcal{D} = \{0, \beta, \pi, \pi + \beta\}$. Consider two consecutive elements o_1 and o_2 in $\mathcal{D}$, in the counterclockwise order, and a point p on the plane. The *stabbing $\mathcal{O}^2_\beta$-wedge associated to* o_1, o_2 with apex p is the open region bounded between two rays emanating from p with orientations o_1 and o_2, respectively. Note that every point p in the plane is the apex of four stabbing $\mathcal{O}^2_\beta$-wedges; top-left, top-right, bottom left, and bottom-right. See Figure 1 (left).

Proposition 1 ([2]) *Let $\mathcal{W}$ be the set of all stabbing $\mathcal{O}^2_\beta$-wedges of the plane containing no elements of P. The $\mathcal{O}^2_\beta$-convex hull of P is* $\mathcal{O}^2_\beta\mathcal{H}(P) = \mathbb{R}^2 - \bigcup_{w \in \mathcal{W}} w$.

2 Computing and maintaining $\mathcal{O}^2_\beta\mathcal{H}(P)$

Based on Proposition 1, in order to compute $\mathcal{O}^2_\beta\mathcal{H}(P)$ we focus on the maximal stabbing $\mathcal{O}^2_\beta$-wedges containing no elements of P.

Moreover, for a point set P and a pair of lines $\mathcal{O}^2_\beta$ we define four staircase polygonal lines, as follows: The *top-right β-staircase* is the following sector of the boundary of the region obtained by removing from the plane all the top-right $\mathcal{O}^2_\beta$-wedges containing no element of P: It starts at the top element of P and ends at the element of P which is the rightmost one with respect to the non-horizontal line in $\mathcal{O}^2_\beta$. In a similar way we can define the top-left, bottom-left, and the bottom-right β-staircases of P.

In the sample $\mathcal{O}^2_\beta$-hull of Figure 1, the dotted lines are the directions of the oriented lines in $\mathcal{O}^2_\beta$ that are used as coordinate axes. Notice that the top-left β-staircase is just a point, and that $\mathcal{O}^2_\beta\mathcal{H}(P)$ is disconnected because of the intersections (the regions bounded by dashed lines) between the top-right and bottom-left β-staircases.

2.1 Maintaining the top-right staircase of P

We now show how to construct and maintain the top-right β-staircase of P as the angle β runs from 0 to π. Start by sorting, in $O(n \log n)$ time, the n points of P from bottom to top, and relabel and place them in this order in a list $\mathcal{L} = \{p_1, \ldots, p_n\}$.

For each p_i, $i = 2, \ldots, n-1$, compute the angles α_i^{a}, α_i^{b}, α_i^{c}, and α_i^{d} as shown in Figure 2. Note that for p_1 and p_n only two of these angles are defined. All these angles can be computed in $O(n)$ time. Notice that for an small enough initial value of β, all the elements of P belong to the top-right β-staircase of P and therefore, $\mathcal{O}^2_\beta\mathcal{H}(P) = P$.

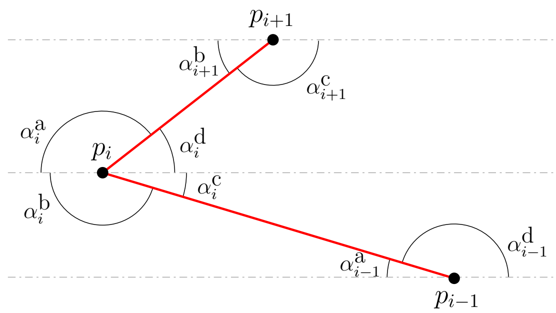

Figure 2: The four angles α_i^{a}, α_i^{b}, α_i^{c}, and α_i^{d} for the point p_i of P.

We observe next that, as the value of β increases, the first element of $\mathcal{L}$ to drop from the list is the p_i with the smallest angle α_i^d. Thus, when β reaches α_i^d, p_i leaves $\mathcal{L}$. Since p_i is no longer considered, we must update the angle of the predecessor p_{i-1} of p_i in $\mathcal{L}$ to be the angle between the horizontal line through p_{i-1} and the segment joining p_{i-1} to p_{i+1}. In a recursive way, if we have removed several elements of $\mathcal{L}$, the next element p_j to be eliminated is that with the smallest α_j^d. This can be obtained in logarithmic time using a priority queue. See Figure 3.

Hence, the total time complexity of calculating and maintaining the top-right β-staircase of P as β increases from 0 to π is $O(n \log n)$ and using linear

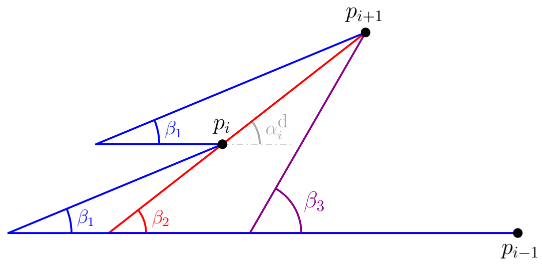

Figure 3: Portion of the top-right staircase for three values of β, before, at, and after the event $\beta = \alpha_i^{\mathrm{d}}$.

space. At the end, when $\beta = \pi$, the only element remaining in $\mathcal{L}$ is the top point p_n.

The top-left, bottom-left, and the bottom-right β-staircases of P can be computed and maintained in a similar way. The four β-staircases of P can be maintained simultaneously, as β goes from 0 to π, in $O(n \log n)$ time and $O(n)$ space.

Lemma 1 *$\mathcal{O}^2_\beta\mathcal{H}(P)$ for $\beta \in [0, \pi]$ can be maintained for $\beta \in [0, \pi]$ in $O(n \log n)$ time and $O(n)$ space.*

Proof. In order to maintain the boundary of $\mathcal{O}^2_\beta\mathcal{H}(P)$ for $\beta \in [0, \pi]$, apart from the four β-staircases of P we also need the sequence of the *overlap-events* which define when overlaps of $\mathcal{O}^2_\beta\mathcal{H}(P)$ finish. Initially, for β slightly greater than zero, consecutive points of P in the y-coordinate order define a very large overlap, whose area will decrease until reaching zero when the corresponding opposite wedges cease to intersect. In order to know when this happens, we need to maintain the current pairs of points which define the opposite wedges determining the overlap, updating them as in Figure 3, and focusing on the two points on rays with the direction of the non-horizontal line in $\mathcal{O}^2_\beta$. See Figure 5. The overlap finishes when β reaches the angle γ between the horizontal and the line through those two points.

The cost of this update is constant, once we know which point is to be changed. Nevertheless, we also need: (i) To maintain the list of the angles γ for all the current overlaps and (ii) To compute the minimum of this list, just to know which is the next overlap-event of ending-overlap. The cost of these updates is at most $O(\log n)$ time per insertion/deletion per point in P, each time such an overlap-event occurs. Since the number of these overlap-events is linear, the total cost is $O(n \log n)$ time. $\square$

Standard techniques (refer to Chapter 4 in [6]) allow to obtain the boundary of $\mathcal{O}^2_\beta\mathcal{H}(P)$ in total $O(n \log n)$ time and $O(n)$ space. Furthermore, this time complexity of the algorithm is optimal, since given $\mathcal{O}^2_\beta\mathcal{H}(P)$ we can compute in linear time

$\mathcal{CH}(\mathcal{O}^2_\beta\mathcal{H}(P)) = \mathcal{CH}(P)$ and the computation of the usual convex hull $\mathcal{CH}(P)$ is in $\Omega(n \log n)$.

From the discussion above we get:

Theorem 2 *$\mathcal{O}^2_\beta\mathcal{H}(P)$ can be computed and maintained for $\beta \in [0, \pi]$ in $\Theta(n \log n)$ time and $O(n)$ space. The numbers of edges and connected components of $\mathcal{O}^2_\beta\mathcal{H}(P)$ for $\beta \in [0, \pi]$ can also be computed and maintained in the same running time and space.*

3 Optimizing the area and perimeter of $\mathcal{O}^2_\beta\mathcal{H}(P)$

Given an angle β, let the polygon $\mathcal{P}(\beta)$ be the one obtained joining counterclockwise consecutive vertices of the four staircases that define $\mathcal{O}^2_\beta\mathcal{H}(P)$.

Following the lines of Bae et al. [3], we express the area of $\mathcal{O}^2_\beta\mathcal{H}(P)$ in terms of the angle β, as

$$\text{area}(\mathcal{P}(\beta)) - \sum_i \text{area}(\triangle_i(\beta)) + \sum_j \text{area}(\varparallelogram_j(\beta)),$$

where: (i) The triangles $\triangle_i$ are defined by a segment joining two consecutive vertices of a β-staircase S of P and the edges joining them along S. (ii) The parallelograms $\varparallelogram_j$ are the overlaps between the boundaries of opposite staircases. See Figure 4.

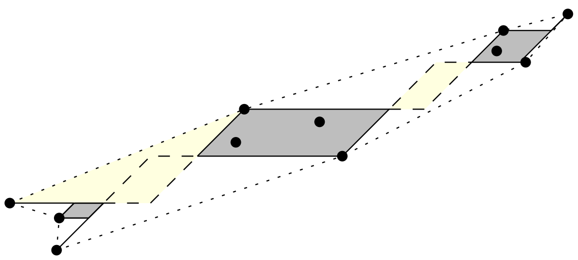

Figure 4: Dotted, the polygon $\mathcal{P}(\beta)$. In dark gray, the area of $\mathcal{O}^2\mathcal{H}_\beta(P)$. In yellow, a triangle and a parallelogram.

Next, we show how to compute each of the three terms in the formula. This allows us to get a general formula, which can be evaluated in each of the intervals $[\beta_i, \beta_{i+1}]$ between two consecutive events, obtaining the value of $\beta \in [\beta_i, \beta_{i+1}]$ which maximizes the area of $\mathcal{O}^2_\beta\mathcal{H}(P)$ in that interval. Note that there is a linear number of these intervals.

3.1 Polygon $\mathcal{P}(\beta)$

Observe that, as β increases from 0 to π, the set of vertices changes a linear number of times. This happens each time a point drops from one of the four staircases of P. Let $\mathcal{A} = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be the set of angles at which the vertices of P drop out from the four staircases of P, $\beta_i < \beta_{i+1}$, $1 \le i \le m-1$.

Since the set of vertices of $\mathcal{P}(\beta)$ remains unchanged for any $\beta \in (\beta_i, \beta_{i+1})$, its area also remains unchanged. Thus, the area of $\mathcal{P}(\beta)$ has to be updated each time β reaches a value in $\mathcal{A}$. Since $\mathcal{A}$ has only a linear number of elements, the area of $\mathcal{P}(\beta)$ has to be updated a linear number of times. Each update can be done in constant time, as it involves the addition or subtraction of the areas of at most two triangles. See Figure 5. A flag will indicate when we have to add or to subtract.

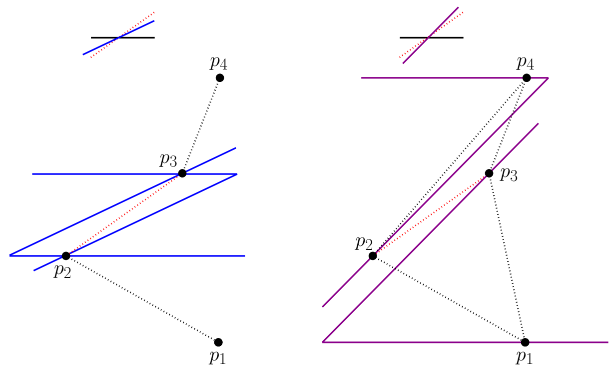

Figure 5: Left: Before the first event, the top-right and the bottom-left β-staircases of P are formed by all the points of P, the top-left β-staircase is the point p_4, and the bottom-right β-staircase is the point p_1. Right: After the first event, p_2 leaves the top-right β-staircase and p_3 leaves the bottom-left β-staircase.

3.2 Triangles $\triangle_i$

Since the number of vertices of $\mathcal{P}(\beta)$ changes only when β reaches a $\beta_i \in \mathcal{A}$, the number of triangles defined by $\mathcal{P}(\beta)$ also changes only when β equals some $\beta_i \in \mathcal{A}$.

Using elementary geometry, it can be checked that the sum of the areas of all the triangles of $\beta_i \in \mathcal{A}$ has the form $c + d\cot(\beta)$: It is sufficient to note that the area of each triangle $\triangle_i$ of $\mathcal{P}(\beta)$ has the form $|c_i \pm d_i \cot(\beta)|$. For example, if $p_i = (x_i, y_i)$ and $p_{i+1} = (x_{i+1}, y_{i+1})$ are consecutive vertices in the counterclockwise order of the top-right β-staircase of P, see Figure 6, then the area of the triangle $\triangle_i$ bounded by (i) the segment joining p_i to p_{i+1}, (ii) the horizontal line through p_i, and (iii) the line with angle β passing through p_{i+1} can be expressed as

$$\text{area}(\triangle_i) = |(x_i - x_{i+1})(y_{i+1} - y_i) + (y_{i+1} - y_i)^2 \cot(\beta)| =$$
$$= |c_i \pm d_i \cot(\beta)|.$$

3.3 Parallelograms $\varparallelogram_j$

Parallelograms arise from overlaps between opposite β-staircases of the P. We need to compute the initial

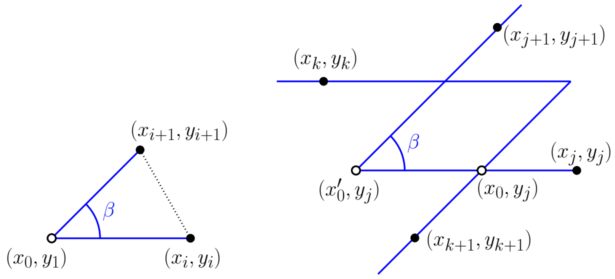

Figure 6: Left: Triangle corresponding to two consecutive points of the top-right β-staircase of P. Right: Parallelogram corresponding to two consecutive points of the top-right β-staircase and two consecutive points of the bottom-left β-staircase.

and final values of β for which each overlap is alive. These overlap events should be merged with the other events, in order to perform a discrete computation updating and computing the maximum values of the variables we want to optimize.

Overlaps can only arise between opposite staircases, that is between the top-right and the bottom-left β-staircases, or between the top-left and the bottom-right β-staircases.

Moreover, as β increases from 0 to π, all the overlaps between the top-left staircases and the bottom-right β-staircases arise after all the overlaps between the top-right and the bottom-left β-staircases. Thus we can process them independently, one after another.

The sum of the areas of these parallelograms can be expressed again as a function of the type $c' + d' \cot(\beta)$. For example, consider a parallelogram ▱$_j$ determined by two consecutive points $p_j = (x_j, y_j)$ and $p_{j+1} = (x_{j+1}, y_{j+1})$ of the top-right β-staircase, together with two consecutive points $p_k = (x_k, y_k)$ and $p_{k+1} = (x_{k+1}, y_{k+1})$ of the bottom-left β-staircase. See Figure 6.

Note that the vertices of the parallelogram ▱$_j$ are not p_j, p_{j+1}, p_k, and p_{k+1}. In fact, the parallelogram ▱$_j$ is the intersection of two triangles of $\mathcal{P}(\beta)$, defined by p_j, p_{j+1}, and p_k, p_{k+1}.

Using elementary geometry, it can be checked that:

As β increases, the number of parallelograms generated by the top-right and the bottom-left β-staircases decreases. We need to compute in advance the events of the ends (and the beginnings) of overlaps, which are exactly the beginning (and the end) events of areas for the top-right and the bottom-left β-staircases the top-left and the bottom-right β-staircases.

Using again a priority queue, we can find the order in which the overlaps disappear in overall $O(n \log n)$ time. When a point defining an overlap changes, we have to update the corresponding area formula. This can be done in constant time.

The discussion above leads to the following result:

Theorem 3 *To compute the angle β such that $\mathcal{O}^2_\beta\mathcal{H}(P)$ has maximum area can be done in $O(n \log n)$ time and $O(n)$ space.*

As for maintaining the perimeter, it is enough to maintain the four staircases and the overlaps.

Thus, we get the following result:

Theorem 4 *To compute the angle β such that $\mathcal{O}^2_\beta\mathcal{H}(P)$ has minimum (non-zero) perimeter can be done in $O(n \log n)$ time and $O(n)$ space.*

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