Simple Alternating Path Problem

JIN AKIYAMA

Department of Mathematics, Tokai University, Hiratsuka, Japan.

JORGE URRUTIA*

Department of Computer Science, University of Ottawa, Ottawa, Ontario, Canada.

Abstract

Let A be a set of $2n$ points in general position on a plane, and suppose n of the points are coloured red while the remaining are coloured blue. An alternating path P of A is a sequence $p_1, p_2,..., p_{2n}$ of points of A such that p_{2i} is blue and p_{2i+1} is red. P is simple if it does not intersect itself. We determine the condition under which there exists a simple alternating path P of A for the case when the $2n$ points are the vertices of a convex polygon. As a consequence an $O(n^2)$ algorithm to find such an alternating path (if it exists) is obtained.

This research was supported under Natural Sciences and Engineering Research Council of Canada (NSERC Canada) grant number A0977.

1. Introduction

Let A be a set of 2n points in general position in the Euclidian plane $\mathbb{R}^2$, and suppose n of the points are coloured red while the remaining are coloured blue. A celebrated Putnam problem posed in 1979 asserts that there are n pairwise disjoint straight line segments matching the red points with the blue points. An extension to higher dimensional cases is discussed in [1].

An alternating path P of A is a sequence $p_1, p_2,...,p_{2n}$ of points of A such that p_{2i-1} is blue and p_{2i} is red, i=0,...n. P is simple if it does not intersect itself.

As a natural extension of the matching assertion, we can ask the following question:

Given an arbitrary collection A of points, does there always exist a simple alternating path P of A?

The configuration of 16 points on a circle shown in Figure 1 shows that the answer to this question is negative.

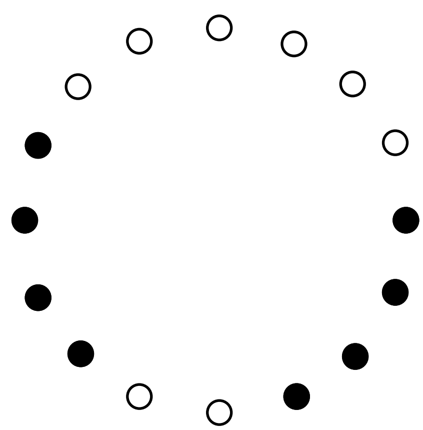

Figure 1

In this paper we will consider collections of points A which form **the vertices of a convex polygon**. We characterize collections of such points for which a simple alternating path P exists. As a consequence, an $O(n^2)$ algorithm to find such a path, if it exists, is obtained. The general case when the elements of A are arbitrarily placed on the plane remains open.

1.1 Terminology and Definitions

Before giving a condition under which such an alternating path exists, let us give a few definitions.

A **word** $S=\{S_0, S_1,....,S_{2n-1}\}$ is a sequence of 2n elements such that n of them are a's and n are b's.

A **circular word** $W=\{S_0, S_1,....,S_{2n-1}\}$ is a word in which S_{2n-1} is followed by S_0,..., etc.

A **subword** W(i,k) of a circular word W is the subsequence $\{S_i, S_{i+1},....,S_{i+2k-1}\}$ of W with 2k elements starting at element S_i, addition taken mod 2n.

A **valid word** W is a word that can be constructed using the following rules:

a) $\emptyset$ is a valid word

b) If W is a valid word, then baW, aWb, bWa and Wab are valid words.

Informally speaking, a word is constructed by alternately adding an a and then a b to the empty word at either end of it.

For example the valid words with two letters are ab and ba; with four letters we have aabb, bbaa, abab, baba, and baab. However abba is not a valid word.

A circular word is a valid circular word if there is an i such that $S_i, S_{i+1},....,S_{i+2n-1}$ is a valid word.

Not all circular words are valid circular words. The reader may check easily that W={aaaabbaaaabbbbbbbb} is not a valid circular word.

2. Main Result

Let $A=\{P_0, P_1,....,P_{2n-1}\}$ be the vertices of a convex polygon such that half of them are coloured a and half are coloured b. Let $W=\{S_0, S_1,....,S_{2n-1}\}$ be the circular word obtained from P_{2n} as follows:

S_i=a if P_i is coloured a, otherwise S_i=b.

Theorem 1: A simple alternating path exists for A if and only if W is a valid circular word.

Before proving Theorem 1 we need the following lemmas:

Lemma 1: Let $□=\{P_{□(0)},....,P_{□(2n-1)}\}$ for A. Then the vertices covered by any initial subpath $\{P_{□(0)},....,P_{□(k)}\}$ of □ cover a subset of vertices of A of the form $\{P_i, P_{i+1},....,P_{i+k-1}\}$.

The proof follows immediately from the definition. (See Figure 2b).

Lemma 2: The subword W(2k) induced in W by the initial segment $P_{□(0)}$, $P_{□(1)}$,....,$P_{□(2k)}$ of □ is a valid word.

Proof: It follows from Lemma 1 and the observation that each time two elements are added to any initial subpath of □, the first one is an a and the second one is a b, thus extending a valid subword of W according to rules (a) and (b).

Theorem 1 now follows immediately

[]

3. The Algorithm

We now present an $O(n^2)$ algorithm to determine if a circular word $W=\{S_0, S_1,....,S_{2n-1}\}$ is a valid circular word.

We will transform the problem of deciding if a circular word W is a valid word into a path problem in a directed graph. The method used will allow us not only to determine if a word is valid or not, but will also tell us all different ways in which a word W can be constructed. This, in turn, will tell us how many alternating paths, if any, exist for A.

Method:

Given W, construct a digraph D(W) with vertices the subwords W(i,k) of W plus the empty word and the total word W as source and sink.

An edge W(i,k)□W(j,k+1) is present in D(W) if W(i,k) can be extended to W(j,k+1) according to rules (a), (b). (See Figure 2a.)

Observations:

The outdegree of the vertices of D(W) (except possibly ∅) and W is at most 4.

Thus |E(D(W))| is $O(n^2)$.

A word is valid if there is a path from ∅ to W in D(W). This can be accomplished in $O(n^2)$.

Example:

W= a a b b a b

S_0 S_1 S_2 S_3 S_4 S_5

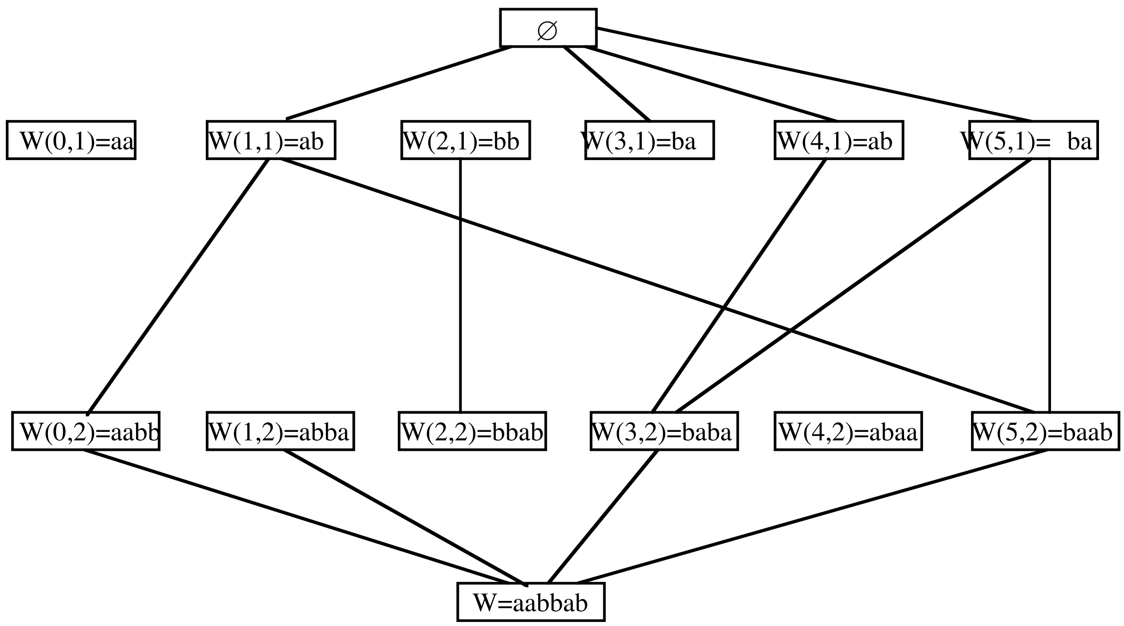

Figure 2a: 5 ways of playing W

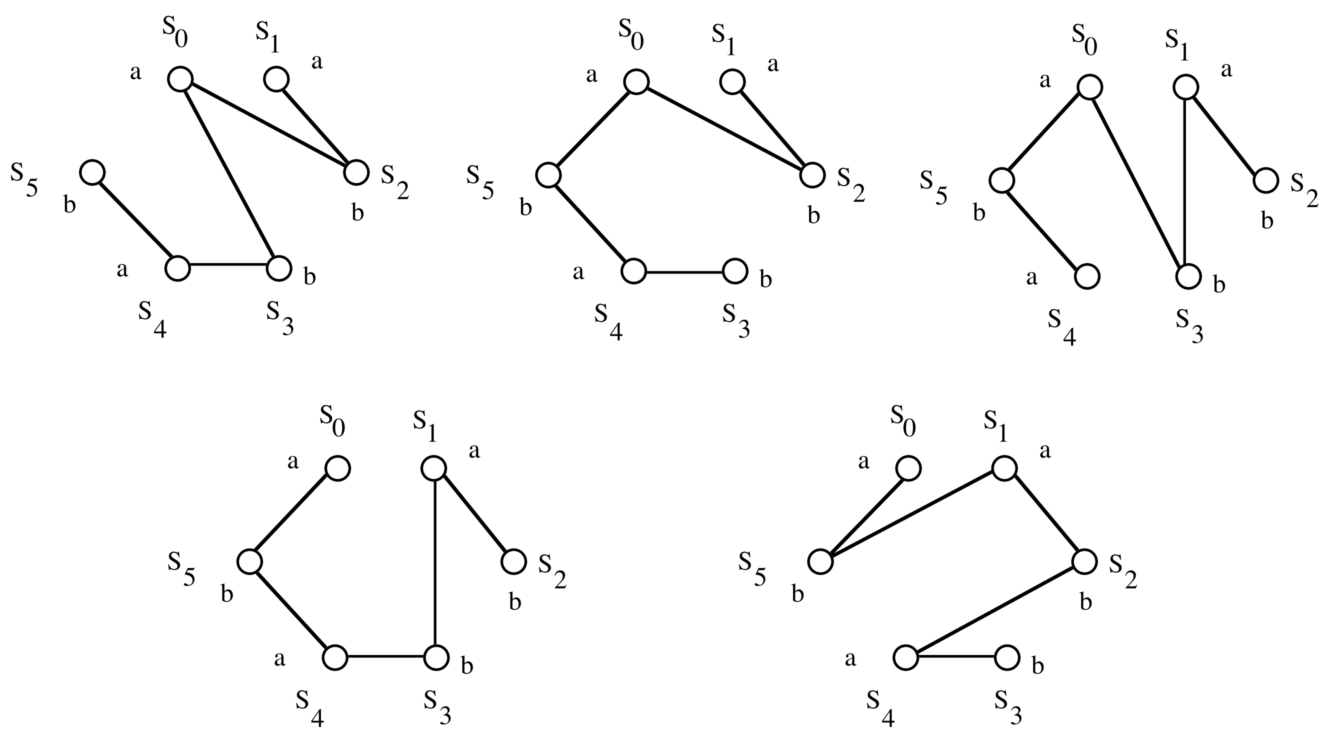

Figure 2b:

Reference

[1] Akiyama, J., and Alon, N. Disjoint Simplices and Geometric Hypergraphs. To appear.