Blocking the k-holes of point sets on the plane

Javier Cano[1], **Alfredo García**[2], **Ferran Hurtado**[3], **Toshinori Sakai**[4], **Javier Tejel**[2], **Jorge Urrutia**[1]

[1] Instituto de Matemáticas, UNAM, Mexico
j_cano@uxmcc2.iimas.unam.mx, urrutia@matem.unam.mx

[2] Departamento de Métodos Estadísticos, IUMA, Universidad de Zaragoza, Spain
olaverri@unizar.es, jtejel@unizar.es

[3] Departament de Matemàtica Aplicada II, UPC, Spain
Ferran.Hurtado@upc.edu

[4] Research Institute of Educational Development, Tokai University, Japan
sakai@tokai-u.co.jp

Abstract. Let P be a set of n points in the plane in general position. A subset h_k of k points of P is called a k-hole if there is no element of P contained in the interior of the convex hull of h_k. A set B of points blocks the k-holes of P if any k-hole of P contains an element of B in its interior. In this paper we establish upper and lower bounds on the sizes of k-hole blocking sets.

Introduction

Let P be a set of n points on the plane in general position. We say that P is in *convex position* if the elements of P are the vertices of a convex polygon. A convex polygon $\mathcal{Q}$ with k vertices is called a k*-gon* of P if all of its vertices belong to P, and $\mathcal{Q}$ is a k*-hole* of P if it contains no element of P in its interior. A point b *blocks* a k-hole $\mathcal{Q}$ of P if it belongs to the interior of $\mathcal{Q}$. A set of points B is a k*-hole blocking set* of P ("k-blocking set of P" for short) if every k-hole of P is blocked by at least one element of B.

The problem of finding point sets that block all the 3-holes of a point set has been studied for some time now. It is known that, if a point set P with n elements has c points on its convex hull, then the 3-holes of P can be blocked with exactly $2n - c + 3$ points; see Katchalski and Meir [**4**], and Czyzowicz, Kranakis and Urrutia [**1**]. Recently, Sakai and Urrutia proved in [**6**] that there are point sets such that $2n - o(n)$ points are necessary to block all their 4-holes. Surprisingly, the problem changes substantially for k-blocking sets, $k \geq 5$. We will show that there are point sets, both in general and in convex position, for which the number of points needed to block their 5-holes is as low as a fifth of the number of triangles in a triangulation of the respective point set. In fact, the number of points needed to block the 5-holes of a point set depends on the geometry of the specific point set, unlike the case of blocking its triangles. For example, not all sets P of n points in convex position require the same number of 5-blockers. It is worth mentioning that the case $k = 2$, *i.e.*, blocking the visibility between pairs of points, has also received attention recently; see [**5**] and the references there.

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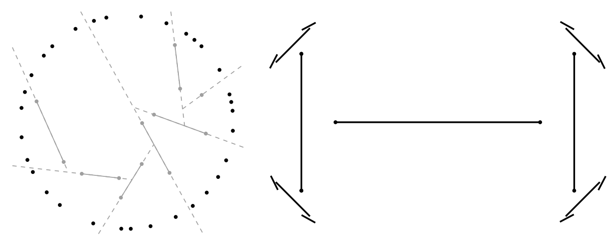

FIGURE 1. (a) Illustration of Theorem 1.1. (b) Point set $\mathcal{X}_4$.

1 Blocking the 5-holes of point sets

In this section we study the problem of blocking the 5-holes of point sets on the plane. We consider first point sets in convex position, and then point sets in general position.

1.1 Point sets in convex position

Theorem 1.1. *Let P a set of n points in convex position. Then any 5-blocking set for P has at least $2\lceil\frac{n}{4}\rceil - 3$ elements.*

Proof. Let B be a 5-blocking set of P with r elements. Let $\mathcal{M}$ be a planar geometric matching of maximum cardinality of the elements of B; that is, a set of disjoint pairs of the elements of B such that the line segments $\{\ell_1, \ldots, \ell_{\lfloor\frac{r}{2}\rfloor}\}$ joining them do not intersect. One at a time, extend them until they hit a line segment or a previously extended segment; some of them might be extended to semi-lines or lines. When r is odd, take a line segment that passes through the unmatched element of B and proceed as before; see Figure 1(a).

This will give us a decomposition of the plane into exactly $\lceil\frac{r}{2}\rceil + 1$ convex regions. Each of these regions can contain at most 4 elements of P; otherwise we would have an unblocked 5-hole. Then $|B| = r \geq 2\left\lceil\frac{n}{4}\right\rceil - 3$. □

Károlyi, Pach and Tóth [**3**] constructed families of point sets which they called *almost convex sets* as follows: Let $\mathcal{R}_1$ be a set of two points in the plane. Assume that we already defined $\mathcal{R}_1, \ldots, \mathcal{R}_j$ such that

(1) $\mathcal{X}_j := \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_j$ is in general position,
(2) the vertex set of the convex hull Γ_j of $\mathcal{X}_j$ is $\mathcal{R}_j$, and
(3) any triangle determined by $\mathcal{R}_j$ contains precisely one point of $\mathcal{X}_j$ in its interior.

Let $z_1, \ldots, z_r$ denote the vertices of Γ_j in clockwise order around Γ_j, and let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$, let ℓ_i denote the line through z_i orthogonal to the bisector of the angle of Γ_j at z_i. Let z_i' and z_i'' be the two points in ℓ_i at distance ε_j from z_i. Now move z_i' and z_i'' away from Γ_j by a distance δ_j in the direction orthogonal to ℓ_i, and denote the resulting points by u_i' and u_i'', respectively.

We can choose ε_j and δ_j to be sufficiently small such that $\mathcal{R}_{j+1} := \{u_i', u_i'' | i = 1, \ldots, r\}$ also satisfies the above conditions. Conditions 1 and 2 are straightforward, so we will verify only the third.

If $a \in \{u_i', u_i''\}$, $b \in \{u_m', u_m''\}$ and $c \in \{u_s', u_s'\}$ are three points of $\mathcal{R}_{j+1}$, for three distinct indices i, m, s, then any point of $\mathcal{X}_{j+1} := R_{j+1} \cup \mathcal{X}_j$ which belongs to the interior of Δabc must coincide with the point of $\mathcal{X}_j$ in the interior of $\Delta z_i z_m z_s$. If we have $a = u_i'$, $b = u_i''$ and $c \in \{u_m', u_m''\}$, with $i \neq m$, then the only point inside Δabc is z_i. Clearly $|\mathcal{X}_m| = 2^{m+1} - 2$ and $|\mathcal{R}_m| = 2^m$, for $m \geq 1$. See Figure 1(b). Now we prove:

Theorem 1.2. *There is a point set P in convex position with $n = 2^m$ that has a 5-blocking set with only $\frac{n}{2} - 2$ elements.*

Proof. Let $P = \mathcal{R}_m$ and $B = \mathcal{X}_{m-2}$. Then $|P| = n$ and $|B| = \frac{n}{2} - 2$. We will show that B is a 5-hole blocking set for P. Suppose that B is not a 5-hole blocking set for P; then we have a 5-hole of P with no point of B in its interior. Take a triangulation of such a 5-hole —it will have 3 triangles of P. By construction, each of them contains exactly one element of $\mathcal{X}_{m-1}$, since $B = \mathcal{X}_{m-1} \setminus \mathcal{R}_{m-1}$. Then these three points have to be elements of $\mathcal{R}_{m-1}$ and they form a triangle contained in the 5-hole. By construction, such a triangle contains precisely one element of $\mathcal{X}_{m-2}$. Now, since $B = \mathcal{X}_{m-2}$, the 5-hole contains an element of B, which is a contradiction. Thus our result follows. □

1.2 Points in general position

Observe that there are point sets in general position for which roughly $\frac{2n}{3}$ points are necessary to block all their 5-holes. Take a set of points P that admits a convex pentagonization of its convex hull, and whose convex hull has five vertices. The number of pentagons in any pentagonization of the convex hull of P is $\lfloor \frac{2n-7}{3} \rfloor$; clearly any 5-blocking set of P has at least $\lfloor \frac{2n-7}{3} \rfloor$ points. We show next that there exist, surprisingly, families of point sets for which all of their 5-holes can be blocked with fewer than $\lfloor \frac{2n-7}{3} \rfloor$ points.

(a) A point set in general position in which $\frac{n}{3} - 2$ points are sufficient and necessary to block all of its convex 5-holes.

(b) The general construction when $k = 11$.

FIGURE 2

Theorem 1.3. *For any m there is a point set P in general position with $n = 3m$ points such that $m - 2$ points are sufficient and necessary to block all the 5-holes of P.*

Proof. Suppose that m is odd. Take a circle $\mathcal{C}$ and m sufficiently small disjoint chords $\{\mathcal{D}_1, \ldots, \mathcal{D}_m\}$ of $\mathcal{C}$ of equal length and evenly placed along $\mathcal{C}$. Each chord $\mathcal{D}_i$ determines a small arc $\mathcal{A}_i$ of $\mathcal{C}$, joining its endpoints. For each chord $\mathcal{D}_i$ select three points of the plane as follows: The first one is the midpoint of $\mathcal{A}_i$, and two points on $\mathcal{D}_i$ are equidistant and close enough to its mid-point so that the shaded region shown in Figure 2(a) is empty. We can think that these 3 points become one *fat point* of an m point set S_m in convex position.

Note that any convex 5-hole of P has at most two vertices in each fat point of S_m. Thus any 5-hole of P contains a point in at least three fat points of S_m. Let P' be the subset of P containing the points in the middle of $\mathcal{A}_i$, $i = 1, \ldots, m$. It is known **[1, 4]** that the set of triangles of P' can be blocked with a set Q_m of $m - 2$ points. It is now easy to see that these points can be chosen in such a way that they also block any triangle containing a point in three different fat vertices of S_m. It is not hard to see that we need at least $m-2$ points to block all the 5-holes of P. For n even, we use a similar construction. Our result follows. □

To finish this section, we prove:

Theorem 1.4. *Let P be a set of points in general position. Then any 5-blocking set of P has at least $2\lceil \frac{n}{9} \rceil - 3$ points.*

As in the proof of Theorem 1.1, we match the points of a 5-blocking set and subdivide the plane into convex regions. The main difference is that we now use a well known result of Harborth [**2**] which states that a point set with ten points always has a 5-hole.

2 Blocking *k*-holes for larger *k*

Now we consider the problem of blocking convex k-holes, $k \geq 6$. Let P be a set of n points in convex position. By a similar argument as in the proof of Theorem 1.1, it can be verified that any k-blocking set for P has at least $2\lceil \frac{n}{k-1} \rceil - 3$ elements. This bound is essentially tight.

To see the tightness for odd k, construct a point set P in the following way: First define integers m and r by $n = \frac{k-1}{2} m + r$, $0 < r < \frac{k-1}{2}$ (here we assume further that $r \neq 0$). We have $m = \lfloor \frac{2n}{k-1} \rfloor$. Let $Q = \{q_1, \ldots, q_{m+1}\}$ be the set of vertices of a regular $(m+1)$-gon, and let C be the circumcircle of this polygon. We replace each q_i by $\frac{k-1}{2}$ points lying on a sufficiently short arc of C (Figure 2(b)), except q_{m+1}, which we replace by r points. Denote by P_i the set of these $\frac{k-1}{2}$ or r points, and let $P = P_1 \cup \cdots \cup P_{m+1}$.

Then any k-hole with vertices in P has vertices in at least three P_i's. Thus the elements of a triangle blocking set for Q (or the points obtained by shifting them slightly if necessary) can block all convex k-holes of P. As in the proof of Theorem 1.3, take a triangle blocking set for Q with $(m+1) - 2 = \lfloor \frac{2n}{k-1} \rfloor - 1$ elements, which will also block all k-holes of P.

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