# Circle Orders, N-gon Orders and the Crossing Number of Partial Orders.

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## Abstract

Let $\mathcal{P}=\{P_1,...,P_m\}$ be a family of sets. A partial order $P(\mathcal{P}, <)$ on $\mathcal{P}$ is naturally defined by the condition $P_i < P_j$ iff $P_i$ is contained in $P_j$. When the elements of $\mathcal{P}$ are disks (i.e. circles together with their interiors), $P(\mathcal{P}, <)$ is called a **circle order**; if the elements of $\mathcal{P}$ are n-polygons, $P(\mathcal{P}, <)$ is called an **n-gon order**. In this paper we study circle orders and n-gon orders. The crossing number of a partial order introduced in [5] is studied here. We show that for every n, there are partial orders with crossing number n. We prove next that the crossing number of circle orders is at most 2 and that the crossing number of n-gon orders is at most 2n. We then produce for every $n\geq 4$ partial orders of dimension n which are not circle orders. Also for every $n\geq 3$, we prove that there are partial orders of dimension 2n+2 which are not n-gon orders. Finally we prove that every partial order of dimension $\leq$ 2n is an n-gon order.

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## 1. Introduction.

The study of simple geometric figures, such as points, circles and polygons has been the source of many interesting and important results since the beginnings of mathematics. The study of incidence relations in collections of points, lines and circles are found in books as far back as the famous *Elements of Euclid*. When dealing with families of simple geometric objects such as circles, polygons, and so on, a basic problem is that of determining the containment relations among the elements of the set. In this paper, we will study partial orders arising from containment problems of families of simple objects on the plane.

Let $\mathcal{P}=\{P_1,...,P_m\}$ be a family of sets. A partial order $P(X, <)$ on a set $X=\{x_1,...,x_m\}$ is said to represent $\mathcal{P}$ if $x_i< x_j$ iff $P_i$ is contained in $P_j$; $P(X, <)$ is called the **containment partial order** of $\mathcal{P}$. Conversely, we say that $\mathcal{P}$ is a **realization** of $P(X, <)$. Every partial order $P(X, <)$ on a set X is the containment partial order of at least one family of sets. This is easily seen by associating with each element $x \in X$ the set $S(x)=\{y \in X: y<x\} \cup \{x\}$. Then $P(X, <)$ is the containment partial order of $\mathcal{P}=\{S(x): x \in X\}$. When the elements of $\mathcal{P}$ are intervals on the real line, the partial orders thus obtained are exactly all partial orders of dimension 2 (See[2]). In [4] it was proved that partial orders of dimension 2n correspond to families of n-boxes in $\mathbb{R}^{2n}$. When the elements of $\mathcal{P}$ are arcs of a circle, we obtain circular permutation graphs (See[7]). In this paper we study partial orders arising from families of circles and convex polygons with n sides, $n\geq3$.

When the elements of $\mathcal{P}$ are disks, that is circles together with their interiors, $P(X, <)$ is called a **circle order**; if the elements of $\mathcal{P}$ are convex n-polygons, $P(X, <)$ is called an **n-gon order**. The problem of characterizing circle orders or n-gon orders seems to be very hard. At present, we do not even know if all partial orders of dimension 3 are circle orders.

In section 2 of this paper, we study the crossing number of a partially ordered set. We show that for every n, there exist partial orders with dimension n and crossing number n-1, $n\geq1$. This solves a problem presented in [5].

In section 3, we study circle orders. We show that the crossing number of circle orders is at most 2.We present a 14 element poset with dimension 4 which is not a circle order. This is the smallest poset known to us that is not a circle order.

In section 4, we show that the crossing number of n-gon orders is at most 2n. We then show partial orders of dimension 2n+2 with (2n +2) + C(2n+2,n+1) elements and crossing number 2n+1 which are not n-gon orders, $n\geq3$ (where C(i,j) is the binomial coefficient i choose j).

Finally in section 5 we prove that all posets with dimension ≤ 2n are n-gon orders, n≥3. For the case n=2 our result easily implies a result by Fishburn and Trotter [3], namely that all posets with dimension ≤ 4 are angle orders. We shall use the terms circle and n-gon to denote either the circle (n-gon) or the circle (n-gon) with its interior. In context no ambiguity will result.

### 1.2 Preliminaries and Definitions.

A binary relation < over a set X defines a **partial order** P(X,<) on X if for any $x,y,z \in X$ it satisfies

(i) $x<y$, $y<z$ implies $x<z$ (transitivity), and

(ii) $x<x$ (antisymmetry).

The partially ordered set P(X,<) is a **linear order** if it also satisfies

(iii) $x<y$ or $y<x$ for all $x,y \in X$.

Let P(X,<) be a poset. A **realizer** of P of size k is a collection of linear orders $\{L_1(X,<_1), L_2(X,<_2), \ldots, L_k(X,<_k)\}$ such that $L_1(X,<_1) \cap L_2(X,<_2) \cap \ldots \cap L_k(X,<_k) = P(X,<)$, where the intersection is defined by $x<y$ iff $x<_i y$ for all i.

It can be easily proved that every poset can be obtained as the intersection of a number of linear orders. Dushnik and Miller [1] define the **dimension** of P, denoted dim P, to be the size of the smallest possible realizer of P. Such a realizer is called a **minimum realizer** of P.

Let $\mathcal{F}=\{f_1,\ldots,f_m\}$ be a family of continuous functions $f_i:[0,1]\to\mathbb{R}$, i=1…m. The family $\mathcal{F}=\{f_1,\ldots,f_m\}$ is called **proper** if the following conditions are satisfied:

a) For any pair of elements $f_i, f_j \in \mathcal{F}$, i≠j, the set of values $S(i,j)=\{p \in [0,1] : f_i(p)=f_j(p)\}$ is finite.

b) $f_i(0)\neq f_j(0)$, $f_i(1)\neq f_j(1)$; i≠j.

c) Each time the graphs of two functions intersect, they cross each other; that is if for some $p \in [0,1]$ we have $f_i(p)=f_j(p)$, then there exists an $\varepsilon>0$ such that for $p-\varepsilon<a<p<b<p+\varepsilon$, $f_i(a)<f_j(a)$ and $f_i(b)>f_j(b)$ or $f_i(a)>f_j(a)$ and $f_i(b)<f_j(b)$.

Informally speaking, a set of functions $\mathcal{F}=\{f_1,\ldots,f_m\}$ is proper if the graphs of any two

elements $f_i$, $f_j$ □ □ intersect a finite number of times and each time they intersect, they cross each other.
Let $X=\{x_1,\ldots,x_m\}$ be a set, and P(X,<) a partial order on X. P(X,<) is called a function order (f-order for short) if there exists a proper set of functions □=$\{f_1,\ldots,f_m\}$ such that $x_i < x_j$ if $f_i(p)<f_j(p)$ for all p □ [0,1]. The set of functions □=$\{f_1,\ldots,f_m\}$ will be called an **f-diagram** for P(X,<). We will also say that P(X,<) **represents** □. It is easy to prove that every poset is an f-order.

## 2. The Crossing Number of a Partial Order.

Given two functions $f_i$, $f_j$ of an f-diagram □=$\{f_1,\ldots,f_m\}$, let S(i,j)={p□ [0,1]: $f_i(p)=f_j(p)$}. The **crossing number** □(□) is now defined as the maximum over the set {|S(i,j)|: $f_i$, $f_j$ □ □}; that is the maximum number of times two elements of □ intersect. The **crossing number** □(P(X,<)) **of a poset** P(X,<) is now defined as min{□(□): □ is an f-diagram for P(X,<)} . Notice that if □(P(X,<))=0, then P(X,<) has an f-diagram □ in which no pair of functions of □ intersect, thus P(X,<) is a linear order. It is also easy to prove that if □(P(X,<))=1, then the dim P(X,<) is 2.

We now consider a special type of f-diagrams in which the curves are piecewise linear. Let $X=\{x_1,...,x_m\}$ and $\{L_1(X,<_1), L_2(X,<_2),..., L_k(X,<_k)\}$ be a realizer of P(X,<). Each linear extension $L_i(X,<_i)$ of P(X,<) induces a permutation □$_i$ on {1,...,m}, i=1,...,k. Consider k vertical lines $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_k$ such that each $\mathbf{L}_i$ is labelled from bottom to top by □$_i$, i=1,...,k. For each j ($1 \le j \le m$) the curve $f_j$ consists of the union of the k line segments which join i on $\mathbf{L}_i$ with i on $\mathbf{L}_{i+1}$, $1 \le i \le k-1$ (See fig. 1). The next result follows easily [5]:

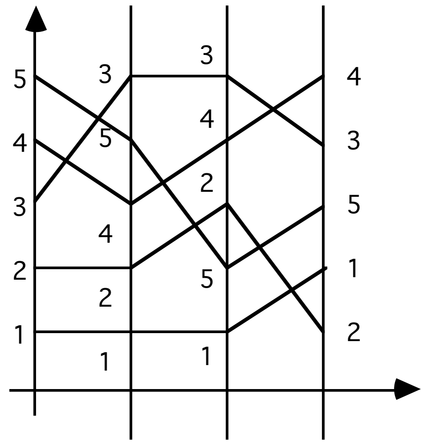


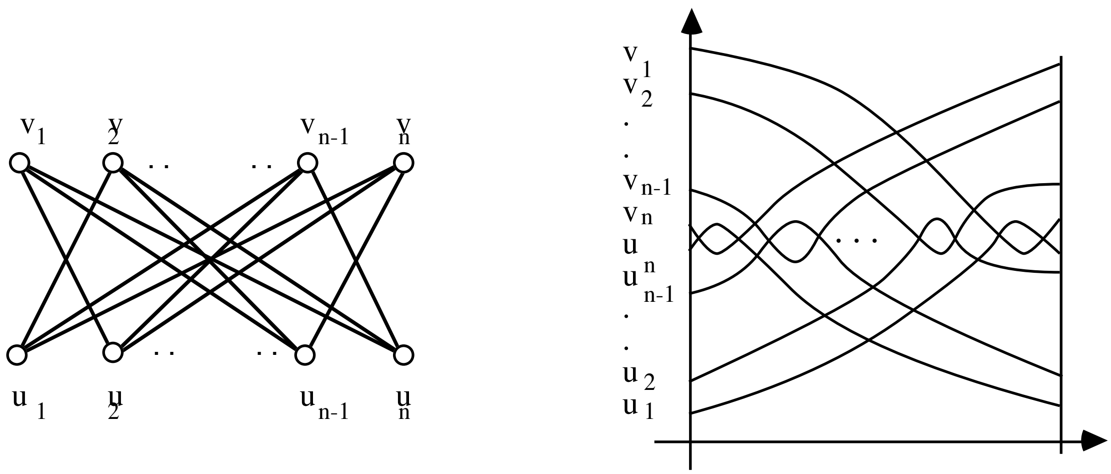

Figure 1.

**Lemma 2.1**: The crossing number □(P(X,<)) of a poset of dimension n is at most n-1.

We now study some properties of a specific poset. Let $□_n$(Y,<) be the poset with elements Y={$u_1$,…,$u_n$ ,$v_1$,…,$v_n$ } such that $u_i$<$v_j$ , i≠j, and all other pairs of elements in Y are not comparable. $□_n$(Y,<) is called the "standard" n-dimension poset. It is well known that the dimension of $□_n$(Y,<) is n. In [5] it was proved that the crossing number of $□_n$(Y,<) is 2, n≥3 (See fig. 2).

In the same paper, the problem of finding posets P with □(P)=n for every n≥3 was posed. In the rest of this section we will show that for every n≥3 there exist posets with crossing number n. Furthermore for every n, we will construct posets with dimension n+1 and crossing number n. Some preliminary results will be required. The following well-known property of $□_n$(Y,<) will be useful:

**i** ) In any linear extension of $□_n$(Y,<), there exists at most one index i such that $u_i$ >$v_i$ and for any k, l≠i, $u_k$ <$v_l$.




(a) (b)

(a) The standard poset $H_n(Y,<)$ on 2n vertices.

(b) An f-diagram for $H_n(Y,<)$ with crossing number 2.

Figure 2.

Let $\mathcal{F}$ be an f-diagram for a poset $P(X,<)$ and $p \in [0,1]$ such that for any $f_i$, $f_j \in \mathcal{F}$, $f_i(p) \neq f_j(p)$, $i \neq j$. Then p induces a **linear extension** $\mathcal{L}(p)$ of P in which $x_i < x_j$ if $f_i(p) < f_j(p)$. When p=0, 1, $\mathcal{L}(0)$ and $\mathcal{L}(1)$ will be called the initial and final linear extensions of $P(X,<)$ with respect to $\mathcal{F}$. The following observation will be used later:

**ii**) Let $\mathcal{F}$ be an f-diagram represented by a poset $P(X,<)$ and $f_i$, $f_j \in \mathcal{F}$ such that $f_i$, $f_j$ represent elements which are not comparable in $P(X,<)$, that is the graphs of $f_i$ and $f_j$ intersect. Then there exists $p \in [0,1]$ such that $f_i(p) < f_j(p)$; moreover p can be chosen in such a way that p induces a linear extension $\mathcal{L}(p)$ of $P(X,<)$.

The next result follows trivially.

**Lemma 2.2**. Let $\mathcal{F} = \{f_1,\ldots,f_n,g_1,\ldots,g_n\}$ be any f-diagram for $\mathcal{H}_n(Y,<)$ in which $u_i$ is represented by $f_i$ and $v_i$ is represented by $g_i$ respectively, $i=1,\ldots,n$. Then there exist n different points $p_1,p_2,\ldots,p_n$ such that $g_1(p_1) < f_1(p_1)$, $g_2(p_2) < f_2(p_2),\ldots$, $g_n(p_n) < f_n(p_n)$.

For every $n \geq 4$, let $\mathcal{H}_n$ be the poset obtained from $\mathcal{H}_n(Y,<)$ as follows: For each subset $S_k$ of $\{1,\ldots,n\}$ with exactly $\lfloor n/2 \rfloor$ and $\lceil (n+1)/2 \rceil$ elements (if n is even both values are the same, if n is odd they are different), insert in $\mathcal{H}_n(Y,<)$ a new element $s_k$ such that $s_k > u_j$, $j \in S_k$, $s_k < v_i$, $i \notin S_k$ and finally if $S_k \supseteq S_{k'}$ then $s_k > s_{k'}$ (See fig. 3 ).

**Lemma 2.3**. The dimension of $\square_n$ is n.

**Proof**: To prove this we notice that $\square_n$ is contained in the poset $2^n$ (under containment) which has dimension n. To see this, let $u_i$ represent the set {i}, $v_i$ be the subsets {1,2,…,n} -{i} and $s_k$ the subset $S_k$ of {1,2,…,n}. The result now follows from the well known result that dim $2^n$ is n.

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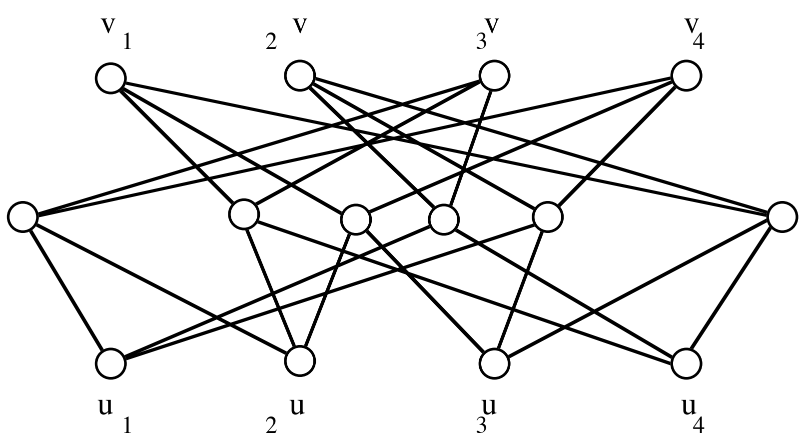


$\square_4$

Figure 3.

**Theorem 2.1**. The crossing number of $\square_n$ is n-1.

**Proof**: Let □' be an f-diagram for $\square_n$ in which $u_i$ is represented by $f_i$, $v_i$ is represented by $g_i$, i=1,…,n and vertices $s_k$ are represented by functions $h_k$. Clearly □' contains an f-diagram □ for $\square_n(Y,<)$ (the one obtained considering the f and g functions only). By Lemma 2.2, there exist n different points $p_1,p_2,\ldots,p_n$ such that $g_1(p_1)< f_1(p_1)$, $g_2(p_2)< f_2(p_2),\ldots, g_n(p_n)< f_n(p_n)$. Let us assume without loss of generality that $p_1<p_2<\ldots<p_n$. Let $S_k$={1,3 ,5,…} and $S_{k'}$={1,…,n } - $S_k$={2,4 ,6,…}. We now prove that the graphs of $h_k$ and $h_{k'}$ intersect in at least n-1 points. To see this, notice that since $S_k$={1, 3 , 5 ,…}, $s_k<v_2,v_4 ,v_6,\ldots$ . Then $h_k(x)<g_i(x)$ for all x□[0,1], i=2,4,6,… . Moreover since $s_k>u_1,u_3 ,u_5,\ldots$ , it follows that $h_k(x)>f_i(x)$ for all x□[0.1], i=1,3,5,… . Similarly, we can prove that $h_{k'}(x)>f_i(x)$, i=2,4,6,… and $h_{k'}(x)<g_i(x)$, i=1,3,5,… . Hence for i=1,3,5,… we have $h_{k'}(p_i)< g_i(p_i)< f_i(p_i)< h_k(p_i)$, i.e. $h_{k'}(p_i)< h_k(p_i)$. Similarly $h_k(p_i)< g_i(p_i)< f_i(p_i)< h_{k'}(p_i)$, i.e. $h_k(p_i)< h_{k'}(p_i)$,

$i=2,4,6,\ldots$ . However since $p_1<p_2<\ldots<p_n$, $h_k$ intersects $h_{k'}$ in each interval $(p_i,p_{i+1})$, $i=1,\ldots,n-1$, i.e. $h_k$ intersects $h_{k'}$ at least n-1 times. Then ( $_n$ )≥ n-1. But by lemma 2.3 dim $_n$=n, and by lemma 2.1 ($_n$)≤n-1. Therefore ($_n$)=n-1.

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**3. Circle Orders.**

Let $X=\{x_1,\ldots,x_m\}$ be a set, and P(X,<) a partial order on X. P(X, <) is called a circle order if there exists a family $=\{P_1,\ldots,P_m\}$ of circles in the plane such that $x_i < x_j$ iff circle $P_i$ is contained inside $P_j$. $=\{P_1,\ldots,P_m\}$ is called a **normal** representation of P(X,<) if int($P_1$)…int($P_m$)≠∅; where int(S) denotes the interior of S.

The next result follows:

**Lemma 3.1**. Any circle order P(X,<) has a normal representation.

**Proof**: Let $=\{P_1,\ldots,P_m\}$ be a circle representation of P(X,<). Let be the maximum distance between the centers of pairs of circles in . For any circle $P_i$ with radius $r_i$ in , let $P_i$(2) be the circle concentric with $P_i$ with radius $r_i$+2. Then $P_i$ is contained inside $P_j$ if and only if $P_i$(2) is contained in $P_j$(2).Hence '=\{$P_1$(2),...,$P_m$ (2)\} is also a circle representation for P(X,<). It is now easy to see that '=\{$P_1$(2),...,$P_m$ (2)\} is a normal representation of P(X,<).

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**Theorem 3.2**. The crossing number of a circle order is at most two.

**Proof**: Let $=\{P_1,\ldots,P_m\}$ be a normal representation of a circle order P(X,<). Let Q be a point in the common intersection of $P_1,\ldots,P_m$ and $L_Q$ a ray starting at Q which does not meet any point in which two circles of $=\{P_1,\ldots,P_m\}$ intersect (See fig. 4). Then using what in topology is known as surgery, cut the plane along $L_Q$ and stretch it so that one side of the cut goes to the Y-axis and the other to the line x=1. Then we obtain an f-diagram  for P(X,<) in which every circle $P_i$ of  is mapped into a function $f_i$ (See fig. 4). Moreover, since any two circles intersect in at most two points, any two functions of  intersect in at most two points. Then ()=2 and (P(X,<))≤2.

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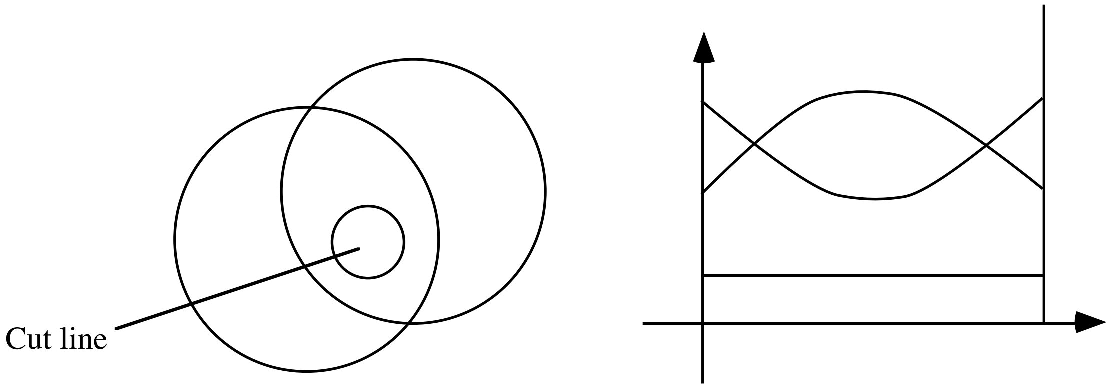


Figure 4.

As a consequence we obtain the following result:

**Theorem 3.3**. For every n≥4 there are partial orders of dimension n which are not circle orders.

**Proof:** The crossing number of $\square_n$ is n-1>2, n≥4.

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For n=4, this proves that $\square_4$ is not a circle order. It is easy to verify that $\square_4$ is critical in the sense that if we delete any element from it, the partial order thus obtained becomes a circle order. Moreover, $\square_4$ is the smallest partial order known to us that is not a circle order. We conjecture that $\square_4$ is the smallest poset which is not a circle order.

**4. N-gon Orders.**

Let $X=\{x_1,\ldots,x_m\}$ be a set, and P(X,<) a partial order on X. P(X, <) is called an n-gon order if there exists a family $\square=\{P_1,...,P_m\}$ of n-polygons on the plane such that $x_i < x_j$ if polygon $P_i$ is contained in $P_j$. We will assume that each time two polygons intersect, they cross each other (as we did in the definition of normal families of functions) and that no vertex belongs to two different elements of $\square$. For the case when the elements of $\square=\{P_1,...,P_m\}$ are triangles, P(X,<) is called a triangle order. We can now prove the following result:

**Theorem 4.1**. The crossing number of n-gon orders is at most 2n.

**Proof:** We will prove our result for the case when P(X,<) is a triangle order. The general case can be easily obtained from this case. We first notice that the boundaries of two different triangles intersect in at most six points. Thus if P(X,<) has a normal representation, the result follows in the same fashion as in theorem 3.2. Suppose then that P(X,<) has a non-normal representation □$=\{P_1,...,P_m\}$. For each triangle $P_i$ □ □ let $P_i$(□)$=\{(x, y)$□$\mathbb{R}^2$: the Euclidian distance between (x, y) and $P_i$ is smaller than or equal to □$\}$. Then for any two triangles $P_i$ , $P_j$ □ □ there exists a number □ large enough that $P_i$(□)□ $P_j$(□)$\neq\emptyset$ and the boundaries of $P_i$(□) and $P_j$(□) intersect in at most six points. Furthermore if $P_i$ is contained in $P_j$ then $P_i$(□) is also contained in $P_j$(□). Let C be a circle of radius □ that contains all the elements of □. Then $P_1$(□)□ $P_2$(□)□ …□ $P_m$(□)$\neq\emptyset$. Moreover, the boundaries of $P_1$(□),$P_2$(□),…,$P_m$(□) form closed curves such that any pair of them intersect in at most six points. Using surgery again, we obtain an f-diagram □ for P(X,<) with crossing number at most six. The proof generalizes easily for n-gons, thus obtaining the desired result.

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Using similar arguments as in Theorem 3.3 we obtain the following result:

**Theorem 4.2**. For every n there are partial orders of dimension 2n+2 which are not n-gon orders.

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**5. N-gon Orders and 2N-dimensional Partial Orders.**

The main objective of this section is to prove the following result:

**Theorem 5.1**. Every poset with dimension $\leq 2n$ is an n-gon poset, $n\geq 3$.

We will need some results before proving this theorem.

Let P(X, <) be a poset of dimension 2 on the set $\{1,\dots,m\}$ and $\{L_1(X, <_1), L_2(X, <_2)\}$ a realizer of P(X, <). Then $L_1(X, <_1)$ and $L_2(X, <_2)$ define two permutations $\prod_1=\{\pi_1(1),\dots,\pi_1(m)\}$ and $\prod_2=\{\pi_2(1),\dots,\pi_2(m)\}$ on $\{1,\dots,m\}$. Using $\prod_1$ and $\prod_2$ we construct a "caged" representation □( P(X, <)) of P(X, <) as follows:

Let □ (h, k) be a rectangle with width h, length k > 1 and sides $S_1$, $S_2$, $S_3$ and $S_4$. Divide each of the two opposite sides $S_1$, $S_3$ of □ (h, k) of length h into m+1 segments of length h/(m+1) using points $p_1,\dots,p_m$ in $S_1$ and $q_1,\dots,q_m$ in $S_3$ (See fig.

5). Place $\pi_1(i)$ on $p_i$ and $\pi_2(i)$ on $q_i$, i=1,…,m. Finally if $\pi_1(i) = \pi_2(j)$ join $p_i$ to $q_j$ by a line segment $L(p_i, q_j)$

(See fig. 5).

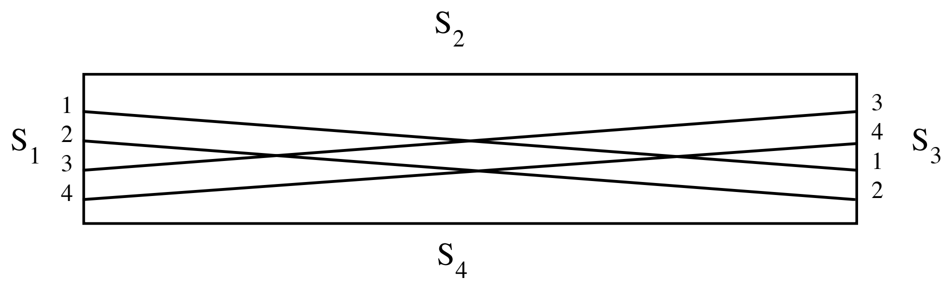


Figure 5.

**Lemma 5.1**. Let p be the intersection point of any two line segments $L(p_i, q_j)$ and $L(p_k, q_l)$ in □(P(X,<)). Then the distance of p to each of $S_1$ and $S_3$ is at least k/m.

**Proof**: The result follows immediately from the fact that the triangles with vertices $\{p_i, p, p_k\}$ and the triangle with vertices $\{q_i, p, q_l\}$ are similar, and that the maximum ratio between the distance of $p_i$ to $p_k$ and the distance between $q_j$ and $q_l$ is at most m.

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For example in fig. 5, the distance between any such p and $S_1$ and $S_3$ is at least k/4.

**Corollary 5.2**. If $k \geq 2m$ then the distance between p and $S_1$ and $S_3$ is at least 2.

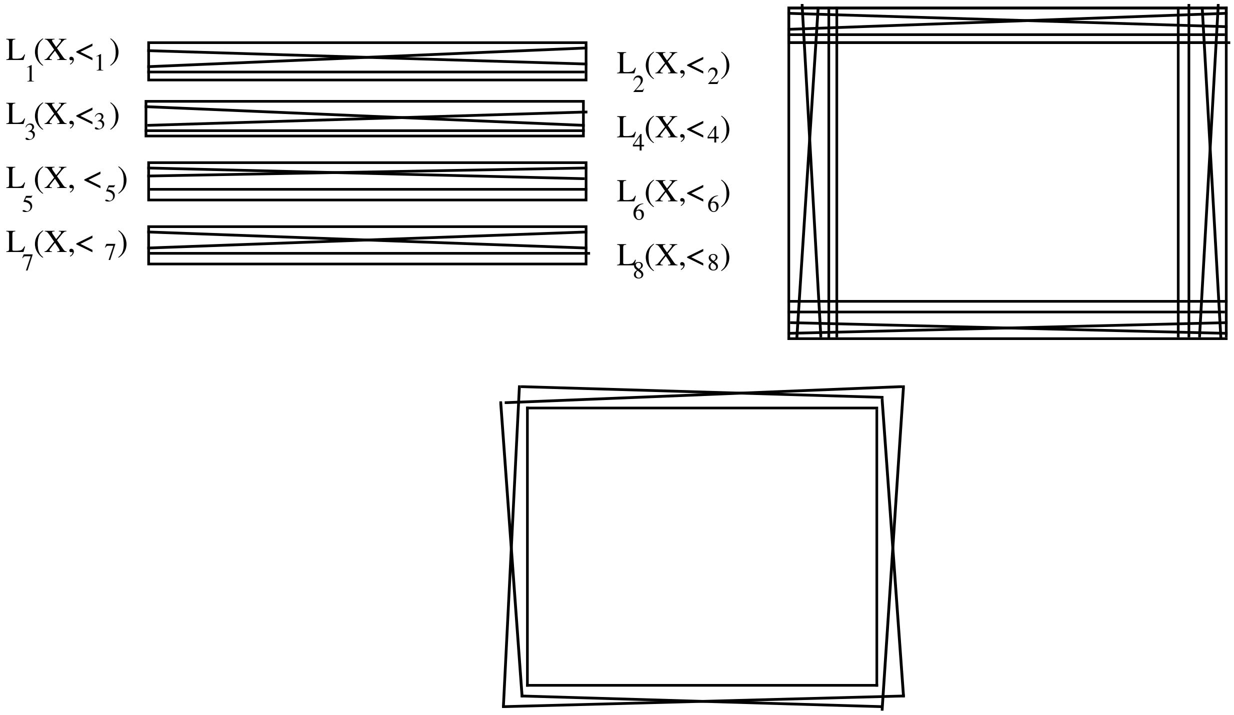


Figure 6.

We are now ready to prove Theorem 5.1.

**Proof**: We carry out the proof for the case n=4. For different values of n the proof easily generalizes. Let $P(X,<)$ be a poset with dimension 8 and $\{L_1(X,<_1),...,L_8(X,<_8)\}$ be a realizer of $P(X,<)$. Assume that $X=\{1,\ldots,m\}$ . Then each $L_i(X,<_i)$ defines a permutation □$_i$ on X, i=1,…,8. Let $P_{i,i+1}(X,<_{i,i+1}) = L_i(X,<_i)$ □ $L_{i+1}(X,<_{i+1})$ and □$(P_{i,i+1}(X,<_{i,i+1}))$ be a caged representation using a rectangle with width 1 and length 2m, i=1,3,5,7. Let □$_4$ be a square on the plane with sides labelled □$_1$,□$_3$, □$_5$,□$_7$ of length 2m. Place □$(P_{i,i+1}(X,<_{i,i+1}))$ in the interior of □$_4$ in such a way that the top of □$(P_{i,i+1}(X,<_{i,i+1}))$ lies on □$_i$, i=1, 3,5,7 (See fig. 6 ).

For each x □ X there is a segment $L_i(x)$ in □$(P_{i,i+1}(X,<_{i,i+1}))$, i=1,3,5,7. If we take
$C(x)=L_1(x)$ □ $L_3(x)$ □ $L_5(x)$□ $L_7(x)$ it divides the plane into two regions one of which (the interior of C(x)) is the interior of an n-gon N(x); x □ X. Since:

$P(X,<)= P_{1,2}(X,<_{1,2})$ □ $P_{3,4}(X,<_{3,4})$ □ $P_{5,6}(X,<_{5,6})$ □ $P_{7,8}(X,<_{7,8})$

it follows now that N(x) is contained in N(y) iff x<y in P(X,<).

When the dimension of $P(X,<)$ is 4, we have two caged representations □( $P_{1,2}(X, <_{1,2})$) and □$P_{3,4}(X, <_{3,4})$ )of $L_1(X,<_1)$ □ $L_2(X,<_2)$ and $L_3(X,<_3)$ □ $L_4(X,<_4)$ respectively. In this case, using only two adjacent sides of our square and properly respacing the elements $p_i$ and $q_i$ in □( $P_{1,2}(X, <_{1,2})$) and □( $P_{3,4}(X, <_{3,4})$), $i=1,\ldots n$, we obtain angle orders [2].

## 6. Conclusions and Open Problems.

The crossing number of circle orders and n-gon orders was studied in this paper. We proved that the crossing number of circle orders is 2. We then proved that for every $n \geq 4$ there are posets with dimension n that are not circle orders. An open question remains:

Are all posets of dimension 3 circle orders?

All posets with crossing number 2 that we have analysed are circle orders. This lead us to formulate the following conjecture:

**Conjecture 1**. All posets with crossing number 2 are circle orders.

Notice that if conjecture 1 is true, this would imply that all posets with dimension 3 are circle orders.

We also proved that the crossing number of n-gon orders is at most 2n. It was then proved that all posets with dimension 2n are n-gon orders and that there are posets of dimension 2n+2 which are not n-gon orders. For posets with dimension 2n+1 we pose the following open problem:

**Problem 1**. Are there posets with dimension 2n+1 which are not n-gon orders?

## References

[1] B. Dushnik and E. Miller. Partially Ordered Sets, *Amer. J. Math.* **6**3 (1941) 600-

610.

[2] P. C. Fishburn. *Interval Orders and Interval Graphs: A Study of Partially Ordered Sets*, John Wiley & Sons (1985).

[3] P. C. Fishburn and W. T. Trotter. Angle Orders, *Order* **1** (1985) 333-343.

[4] M. C. Golumbic. Containment and Intersection Graphs, I.B.M. Scientific Center, T.R. 135, (1984).

[5] M. C. Golumbic, D. Rotem, and J. Urrutia. Comparability Graphs and Intersection Graphs, *Discrete Mathematics* **43** (1983) 37-46.

[6] T. Hiraguchi. On the dimension of Partially Ordered Sets, *Sci. Rep. Kanazawa Univ.* **1**. (1951) 77-94. MR 17, p 19.

[7] D. Rotem and J. Urrutia. Circular Permutation Graphs, *Networks* 12 (1982), 429-438.