A Combinatorial Result About Points and Balls in Euclidean Space

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A theorem of Neumann-Lara and Urrutia [3] is generalized from the plane to arbitrary n-dimensional Euclidean space $\mathbb{R}^n$, solving Problem 2 of [3]. By an n-*ball* we mean a set of the form $\{(x_1, x_2,...,x_n) \in \mathbb{R}^n: (x_1-a_1)^2 + (x_2-a_2)^2 + ... + (x_n-a_n)^2 \leq r\}$, where $(a_1,a_2,...,a_n) \in \mathbb{R}^n$ and $r > 0$.

<u>Theorem 1</u>. For each $n \geq 1$ there is $c_n > 0$ such that for any finite set $X \subseteq \mathbb{R}^n$ there is $A \subseteq X$, $|A| \leq 1/_2 (n+3)$, having the following property: if $B \supseteq A$ is an n-ball, then $|B \cap X| \geq c_n|X|$.

This theorem is seen to be optimal in quite a strong way. Let X be any finite set of points on the *moment curve* $\square(t) = (t, t_2, t_3,...,t_n)$, $|X| = m \geq n+1$. Then X is the set of vertices of a convex polyhedron (known as *the cyclic* n-*polytope with* m *vertices*) and every $_2 (n+1)$ -element subset A X is the set of vertices of one of its faces. (See sections 4.7 and 7.4 of [2].) Clearly then, for each such A there is an n-ball B such that $B \cap X = A$.

The following notation will be used. For a set S, $\square_n(S)$ is the set of n-element subsets of S. If $A \subseteq \mathbb{R}^n$, then conv A is the convex hull of A.

<u>Lemma 2</u>. Let $Y \in \square_{n+3}(\mathbb{R}^n)$. Then there is $A \subseteq Y$, $|A| = {}^1/_2 (n+3)$, such that for any n-ball $B \supseteq A$, $(Y \setminus A) \cap B \neq \emptyset$.

<u>Proof</u>: There exist disjoint $A_1, A_2 \subseteq Y$ such that $|A_1| = |A_2| = 1/_2 (n+3)$ and conv $A_1 \cap$ conv $A_2 \neq \emptyset$. The argument for obtaining A_1 and A_2 is essentially in [1] and [4]. Let $Y = \{y_2, y_2, ...,y_{n+3})$, and then let $\underline{Y} = \{\underline{y}_2, \underline{y}_2, ...,\underline{y}_{n+3}) \subseteq \mathbb{R}^2$ be its Gale transform. (Here we are assuming, without loss of generality, that $\mathbb{R}^n$ is the affine span of Y.) For some $y_i \in Y$, the line $\square$ in $\mathbb{R}^n$ through $\underline{y}_i$ and the origin divides $\mathbb{R}^n$ into two open half-planes P_1, P_2 such that $|P_1 \cap \underline{Y}|$, $|P_2 \cap \underline{Y}| \leq 1/_2 (n+3)$. Let $C_1, C_2, Z \subseteq Y$ be such that $\underline{C}_1 = P_1 \cap \underline{Y}$, $\underline{C}_2 = P_2 \cap \underline{Y}$ and $Z = \square \cap \underline{Y}$. By Lemma 1 of [4], conv $(C_1 \cup Z_1) \cap$ conv $(C_2 \cup Z_2) \neq \emptyset$ whenever $Z_1 \cup Z_2 = Z$. But this implies conv $C_1 \cap$ conv $C_2 \neq \emptyset$. So just let A_1, $A_1 \subseteq Y$ be disjoint sets such that $C_1 \subseteq A_1$, $C_2 \subseteq A_2$ and $|A| = |A_1| = [1/_2 (n+3)]$.

We now claim that either $A = A_1$ works or $A = A_2$ works.

In order to derive a contradiction, let $a \in \text{conv } A_1 \cap \text{conv } A_2$, and let B_1, B_2 be n-balls for which $A_1 \subseteq B_1$, $A_2 \subseteq B_2$ and $B_1 \cap A_2 = \emptyset = B_2 \cap A_1$. Clearly $B_1 \cap B_2 \neq \emptyset$ since $a \in B_1 \cap B_2$, and also $B_1 \backslash B_2 \neq \emptyset \neq B_2 \backslash B_1$. Therefore, there is a unique hyperplane h such that $h \cap \partial B_1 = h \cap \partial B_1 = h \cap B_1$ (where ∂B_i denotes the boundary of B_i). Let $\mathcal{H}_1, \mathcal{H}_2$ be the closed half-spaces such that $\mathcal{H}_1 \cap \mathcal{H}_2 = h$, $B_1 \backslash B_2 \subseteq \mathcal{H}_1$ and $B_2 \backslash B_1 \subseteq H_2$. Then $a \in \mathcal{H}_1 \cap \mathcal{H}_2 = h$, so there must be some $b \in \mathcal{H}_1 \cap h$. But then $b \in B_2$, which is a contradiction.

A simple coumting argument allows us to deduce Theorem 1 from Lemma 2. This is abstracted in the next lemma.

<u>Lemma 3</u>. Let S be an infinite set, $\mathcal{B}$ a collection of subsets of S, and r and m positive integers with $r \geq m+2$. Suppose that for each $Y \in \mathcal{P}_r(S)$ there is $A \in \mathcal{P}_m(Y)$ such that whenever $A \subseteq B \in \mathcal{B}$, then $(Y \backslash A) \cap B \neq \emptyset$. Let $c = (m!\,(r-m-1)!)/r!$. Then for any sufficiently large, finite $X \subseteq S$ there is $A \in \mathcal{P}_m(X)$ such that whenever $A \subseteq B \in \mathcal{B}$, then $|B \cap X| > c\,|X|$.

<u>Proof</u>: First notice that

$$(*) \qquad c < 1 - \left[1 - \frac{1}{\binom{r}{m}} \right]^{\frac{1}{r-m}}$$

To see why, let $b = 1/\binom{r}{m}$ so that $0 < c < b < 1$. Then (*) holds iff $(1-b)^{1/b} < (1-c)^{1/c}$, and the latter inequality holds since $(1-x)^{1/x}$ is a decreasing function on (0, 1).

For integers $t \geq \frac{r}{1-c}$ consider $X \in \mathcal{P}_t(S)$. For such an X, there are sets $A \in \mathcal{P}_m(X)$ and $\mathcal{Y} \subseteq \mathcal{P}_r(X)$ such that $|\mathcal{Y}| \geq \frac{\binom{t}{r}}{\binom{t}{m}}$ and for each $Y \in \mathcal{Y}$, A is as in the hypothesis of the lemma. We claim that this is the desired A if t is large enough.

For suppose that for arbitrarily large $t \geq \frac{r}{1-c}$ there are X, A and $\mathcal{Y}$ as above such that for some $B \in \mathcal{B}$, $B \supseteq A$ and $|B \cap X| \leq ct$. The number of sets $Y \in \mathcal{P}_r(X)$ for which $Y \cap B = A$ is at least $\binom{t-ct}{r-m}$. No such Y is in $\mathcal{A}$; therefore,

$$f(t) \equiv \frac{\binom{t}{m}}{\binom{t}{r}} \left[\binom{t-m}{r-m} - \binom{t-ct}{r-m} \right] \geq 1$$

For all $t \geq \frac{r}{1-c}$, $f(t) \leq g(t)$, where

$$g(t) \equiv \frac{t^m}{m} \cdot \frac{r!}{(k-r)!} \left[\frac{(t-m)^{r-m}}{(r-m)!} - \frac{(t-ct)^{r-m} - (r-m)^2 (t-ct)^{r-m-1}}{(r-m)!} \right]$$

Then $\lim_{t \to \infty} g(t) = \binom{r}{m} \left[1 - (1-c)^{r-m} \right] \geq 1$. Therefore

$$c \geq 1 - \left[1 - \frac{1}{\binom{r}{m}} \right]^{\frac{1}{r-m}}$$, contradicting (*).

The above argument can be used to show that $c_2 > \frac{1}{30}$, improving the constant in [3].

Theorem 1 has several generalizations. We mention just one of them.

Theorem 4. For each $m \geq \lceil 1/_2 (n+3) \rceil$ there is $c_{n,m} > 0$ such that for any finite $X \subset \mathbb{R}^n$, $|X| \geq m$, there is $A \in \mathcal{P}_m(X)$ having the following property: if B is an n-ball and $|A \cap B| \geq \lceil 1/_2 (n+3) \rceil$, then $|B \cap X| \geq c_{n,m}|X|$.

This theorem is a consequence of Lemma 5 below (which is the analogue of Lemma 2) and a version of Lemma 3 whose statement and proof can easily be supplied.

Let $R_s(t)$ be the Ramsey number defined as follows: $R_s(t)$ is the least r such that whenever $|Y| \geq r$ and $\mathcal{P}_s(Y) = P_1 \cup P_2$, then there is $W \in \mathcal{P}_t(Y)$ such that either $\mathcal{P}_s(W) \subseteq P_1$, or $\mathcal{P}_s(W) \subseteq P_2$.

Lemma 5. Let $m \geq s = \lceil 1/_2 (n+3) \rceil$, let $t > m/c_n$ be an integer (c_n is from Theorem 1), and let $r = R_s(t)$. Suppose $Y \in \mathcal{P}_r(\mathbb{R}^n)$. Then there is $A \in \mathcal{P}_m(Y)$ such that if B is an n-ball and □B □ A□≥ s , then (Y \ A) □ B ≠ Ø.

<u>Proof</u>: Let P = {Z □ $□_s$(Y) : for each n-ball B ⊇ Z , □B □ Y□≥ c_nt} . By Ramsey's Theorem there is W □ $□_t$(Y) such that $□_s$(W) □ P or $□_s$(W) □ P ≠ Ø. By Theorem 1, $□_s$(W) □ P ≠ Ø ; hence, $□_s$(W) □ P . Any A □ $□_m$(W) will do, for □B □ A□≥ s implies
□B □ Y□≥ c_nt > m =□A□.

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References

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