On Edge-Disjoint Empty Triangles of Point Sets *

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Abstract

Let P be a set of points in the plane in general position. Any three points $x, y, z \in P$ determine a triangle $\Delta(x, y, z)$ of the plane. We say that $\Delta(x, y, z)$ is empty if its interior contains no element of P. In this paper we study the following problems: What is the size of the largest family of edge-disjoint triangles of a point set? How many triangulations of P are needed to cover all the empty triangles of P? We also study the following problem: What is the largest number of edge-disjoint triangles of P containing a point q of the plane in their interior? We establish upper and lower bounds for these problems.

1 Introduction

Let P be a set of n points in the plane in general position. A geometric graph on P is a graph G whose vertices are the elements of P, two of which are adjacent if they are joined by a straight line segment. We say that G is plane if it has no edges that cross each other. A triangle of G consists of three points $x, y, z \in P$ such that xy, yz, and zx are edges of G; we will denote it as $\Delta(x, y, z)$. If in addition $\Delta(x, y, z)$ contains no elements of P in its interior, we say that it is empty.

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In a similar way, we say that if $x, y, z \in P$, then $\Delta(x, y, z)$ is a *triangle* of P, and that xy, yz, and zx are the edges of $\Delta(x, y, z)$. If $\Delta(x, y, z)$ is empty, it is called a 3-*hole* of P. A 3-hole of P can be thought of as an empty triangle of the complete geometric graph $\mathcal{K}_{\mathcal{P}}$ on P. We remark that $\Delta(x, y, z)$ will denote a triangle of a geometric graph, and also a triangle of a point set.

A well-known result in graph theory says that for $n = 6k + 1$, or $n = 6k+3$, the edges of the complete graph K_n on n vertices can be decomposed into a set of $\binom{n}{2}/3$ edge-disjoint triangles. These decompositions are known as Steiner triple systems [23]; see also Kirkman's schoolgirl problem [17, 22]. In this paper, we address some variants of that problem, but for geometric graphs.

Given a point set P, let $\delta(P)$ be the size of the largest set of edge-disjoint empty triangles of P. It is easy to see that for point sets in convex position with $n = 6k + 1$ or $n = 6k + 3$ elements, $\delta(P) = \binom{n}{2}/3$. Indeed any triangle of P is empty, and the problem is the same as that of decomposing the edges of the complete geometric graph $\mathcal{K}(P)$ on P into edge-disjoint triangles. On the other hand, we prove that for some point sets, namely Horton point sets, $\delta(P)$ is $O(n \log n)$.

We then study the problem of covering the empty triangles of point sets with as few triangulations of P as possible. For point sets in convex position, we prove that we need essentially $\binom{n}{3}/4$ triangulations; our bound is tight. We also show that there are point sets P for which $O(n \log n)$ triangulations are sufficient to cover all the empty triangles of P for a given point set P.

Finally, we consider the problem of finding a point q not in P contained in the interior of many edge-disjoint triangles of P. We prove that for any point set, there is a point $q \notin P$ contained in at least $n^2/12$ edge-disjoint triangles. Furthermore, any point in the plane, not in P, is contained in at most $n^2/9$ edge-disjoint triangles of P, and this bound is sharp. In particular, we show that this bound is attained when P is the set of vertices of a regular polygon.

1.1 Preliminary work

The study of counting and finding k-holes in point sets has been an active area of research since Erdős and Szekeres [11, 12] asked about the existence of k-holes in planar point sets. It is known that any point set with at least ten points contains 5-holes; e.g. see [14]. Horton [15] proved that for $k \geq 7$ there are point sets containing no k-holes. The question of the existence of 6-holes remained open for many years, but recently Nicolás [19] proved that any point set with sufficiently many points contains a 6-hole. A second proof of this result was subsequently given by Gerken [13].

The study of properties of the set of triangles generated by point sets on the plane has been of interest for many years. Let $f_k(n)$ be the minimum number of k-holes that a point set has. Clearly a point set has a minimum of $f_3(n)$ empty triangles. Katchalski and Meir [16] proved that $\binom{n}{2} \leq f_3(n) \leq cn^2$ for some $c < 200$; see also Purdy [21]. Their lower bounds were improved by Dehnhardt [9] to $n^2 - 5n + 10 \leq f_3(n)$. He also proved that $\binom{n-3}{2} + 6 \leq f_4(n)$. Point sets with few k-holes for $3 \leq k \leq 6$ were obtained by Bárány and Valtr [2]. The interested reader can read [18] for a more accurate picture of the developments in this area of research.

Chromatic variants of the Erdős-Szekeres problem have recently been studied by Devillers, Hurtado, Károly, and Seara [10]. They proved among other results that any bi-chromatic point set contains at least $\frac{n}{4} - 2$ compatible monochromatic empty triangles. Aichholzer *et al.* [1] proved that any bi-chromatic point set always contains $\Omega(n^{5/4})$ empty monochromatic triangles; this bound was improved by Pach and Tóth [20] to $\Omega(n^{4/3})$.

2 Sets of edge-disjoint empty triangles in point sets

Let P be a set of points in the plane, and $\delta(P)$ the size of the largest set of edge-disjoint empty triangles of the complete graph $\mathcal{K}(P)$ on P. In this section we study the following problem:

Problem 1. How small can $\delta(P)$ be?

We show that if P is a Horton set, then $\delta(P)$ is $O(n \log n)$. On the other hand, it follows directly from Theorem 7 that if P is the set of vertices of a regular polygon then $\delta(P)$ is at least $\frac{n^2}{9} - n$.

For any integer $k \geq 1$, Horton [15] recursively constructed a family of point sets H_k of size 2^k as follows:

(a) $H_1 = \{(0,0),(1,0)\}$.

(b) H_k consists of two subsets of points H^-_{k-1} and H^+_{k-1} obtained from H_{k-1} as follows: If $p = (i,j) \in H_{k-1}$, then $p' = (2i,j) \in H^-_{k-1}$ and $p'' = (2i+1, j+d_k) \in H^+_{k-1}$. The value d_k is chosen large enough such that any line ℓ passing through two points of H^+_{k-1} leaves all the points of H^-_{k-1} below it; see Figure 1.

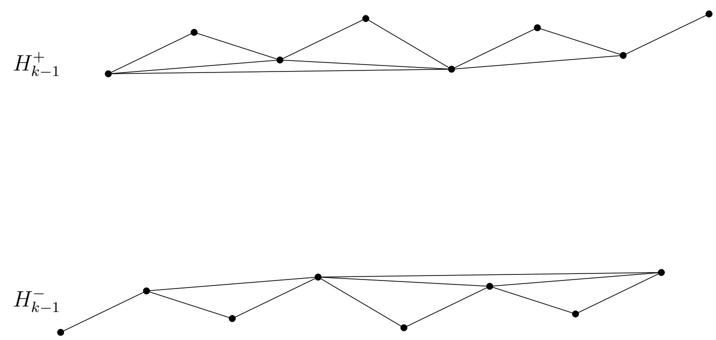

Figure 1: H_4. The edges of H_3^+ (resp. H_3^-) visible from below (resp. above), are shown.

We say that a line segment pq joining two elements p and q of H_k is *visible from below* (resp. *above*) if there is no point of H_k below it (resp. above it); that is there is no element r of H_k such that the vertical line through r intersects pq above r (resp. below r). Let $B(H_k)$ be the set of line segments of H_k visible from below. The following result which we will use later was proved by Bárány and Valtr in [2]; see also [3]:

Lemma 1. $|B(H_k)| = 2^{k+1} - (k+2)$.

The following result is proved in [3] by using this lemma:

Theorem 1. *For every $n = 2^k$, $k \geq 1$, there is a point set (namely H_k) such that there is a geometric graph on H_k with $\binom{n}{2} - O(n \log n)$ edges with no empty triangles.*

In other words, it is always possible to remove $O(n \log n)$ edges from the complete graph $\mathcal{K}_{H_k}$ in such a way that the remaining graph contains no empty triangles. The main idea is that by removing from $\mathcal{K}_{H_k}$ all the edges of H^+_{k-1} (respectively H^-_{k-1}) visible from below (respectively above), no empty triangle remains with vertices in both H^+_{k-1}, and H^-_{k-1}.

Observe now that if a geometric graph has k edge-disjoint empty triangles, then we need to take at least k edges away from G for the graph that remains to contain no empty triangles. It follows now that the complete graph $\mathcal{K}_{H_k}$ has at most $O(n \log n)$ edge-disjoint empty triangles. Thus we have proved:

Theorem 2. *There is a point set, namely H_k, such that any set of edge-disjoint empty triangles of H_k contains at most $O(n \log n)$ elements.*

Clearly for any point set P, the size of the largest set of edge-disjoint triangles of P is at least linear. We conjecture:

Conjecture 1. *Any point set P in general position always contains a set with at least $O(n \log n)$ edge-disjoint empty triangles.*

3 Covering the triangles of point sets with triangulations

An empty triangle t of a point set P is covered by a triangulation T of P if one of the faces of T is t. In this section we consider the following problem:

Problem 2. How many triangulations of a point set are needed such that each empty triangle of P is covered by at least one triangulation?

This problem, which is interesting on its own right, will help us in finding point sets for which $\delta(P)$ is large. We start by studying Problem 2 for point sets in convex position, and then for point sets in general position.

3.1 Points in convex position

All point sets P considered in this subsection will be assumed to be in convex position, and their elements labeled $\{p_0, \ldots, p_{n-1}\}$ in counter-clockwise order around the boundary of $\mathrm{CH}(P)$. Since any triangulation of a point set of n points in convex position corresponds to a triangulation of a regular polygon with n vertices, solving Problem 2 for point sets in convex position is equivalent to solving it for point sets whose elements are the vertices of a regular polygon. Suppose then that P is the set of vertices of a regular polygon, and that c is the center of such a polygon.

A triangle is called an *acute* triangle if all of its angles are smaller than $\frac{\pi}{2}$. We recall the following result in elementary geometry given without proof.

Observation 1. *A triangle with vertices in P is acute if and only if it contains c in its interior.*

The following result is relatively well known:

Lemma 2. *Let P be the set of vertices of a regular n-gon Q, and c the center of Q. Then:*

- *If n is even, c is contained in the interior of $\frac{1}{4}\left[\binom{n}{3} - \frac{n(n-2)}{2}\right]$ acute triangles of P.*

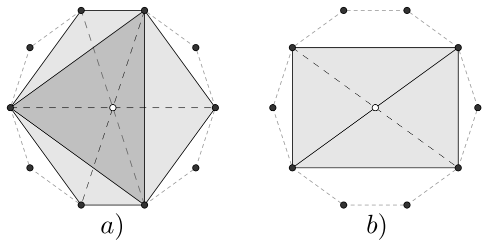

Figure 2: a) Constructing $t_4(i,j,k)$, and b) pairing triangles sharing an edge which contains c in the middle.

- *If n is odd, c is contained in* $\left[\binom{n}{3} - \frac{n(n-1)(n-3)}{8}\right] = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-1)}{2}\right]$ *acute triangles of P.*

Let $f(n) = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-2)}{2}\right]$ for n even, and $f(n) = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-1)}{2}\right]$ for n odd. We now prove:

Theorem 3. *$f(n)$ triangulations are always sufficient, and always necessary, to cover all the triangles of a regular polygon.*

Proof. Suppose first that n is even. For each vertex p_i of P, let $\alpha(p_i) = p_{i+\frac{n}{2}}$ be the antipodal vertex of p_i in P, where addition is taken mod n. Suppose that $\Delta(p_i, p_j, p_k)$ is an acute triangle of P (i.e. it contains c in its interior), $i < j < k$. Let $t_4(i,j,k)$ be the following set of four triangles:

$$t_4(i,j,k) = \{\Delta(p_i,p_j,p_k), \Delta(\alpha(p_i),p_j,p_k), \Delta(p_i,\alpha(p_j),p_k), \Delta(p_i,p_j,\alpha(p_k))\};$$

see Figure 2 a).

It is easy to see that all the triangles of P except those that have a right angle are in

$$\bigcup t_4(i,j,k),$$

where i,j,k range over all triples such that $\Delta(p_i,p_j,p_k)$ contains c in its interior.

On the other hand, it is easy to see that if a triangle t of P contains c in the middle of one of its edges (clearly t is a right triangle), this edge joins two antipodal vertices of P; see Figure 2 b). Thus we have exactly

$$\frac{n}{2} \times (n-2)$$

such triangles. It is easy to find

$$\frac{n(n-2)}{4}$$

triangulations of P such that each of them cover two of these triangles. Since each triangulation of P contains exactly one acute triangle of P or two triangles sharing an edge that contains c at its middle point, it follows that

$$\frac{1}{4}\left[\binom{n}{3} - \frac{n(n-2)}{2}\right] + \frac{n(n-2)}{4} = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-2)}{2}\right]$$

triangulations are necessary and sufficient to cover all the triangles of P. To show that this number of triangulations are needed, we point out that any two acute triangles of P cannot belong to the same triangulation (note that they intersect at c). Moreover these triangulations are different from those containing right triangles. Our result follows.

A similar argument follows for n odd, except that some extra care has to be paid to the way in which we group the non-acute triangles of P around the acute triangles of P. □

Thus the number of triangulations needed to cover all the triangles of P is asymptotically $\binom{n}{3}/4$. The next result follows trivially:

Corollary 1. *Let P be a set of n points in convex position, and p any point in the interior of $CH(P)$. Then p belongs to the interior of at most $\frac{\binom{n}{3}}{4} + O(n^2)$ triangles of P.*

3.2 Covering the empty triangles on the Horton set

We will now show that all the empty triangles in H_k can be covered with $O(n \log n)$ triangulations. The bound is tight.

Consider an edge e of H_k that is visible from below, and a vertical line ℓ that intersects e at a point q in the interior of e. The depth of e is the number of edges of H_k, visible from below, intersected by ℓ below q. It is not hard to see that the maximal depth of an edge of H_k visible from below is at most $\log n - 1$, and that this bound is tight; see Figure 3. Moreover, it is easy to see that the union of all edges of H_k with the same depth is an x-monotone path. Now we can prove:

Theorem 4. $\Theta(n \log n)$ *triangulations of H_k are necessary and sufficient to cover the set of empty triangles of H_k.*

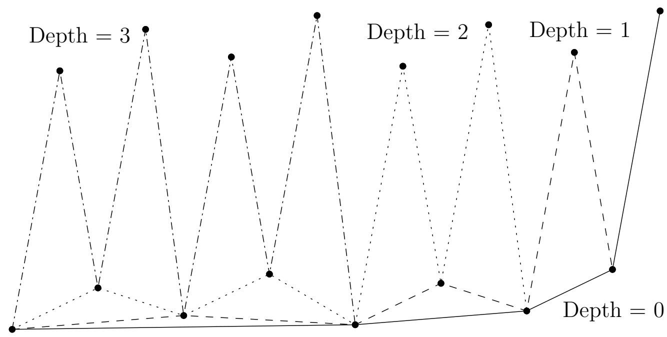

Figure 3: The depth of an edge.

Proof. Consider the sets H^+_{k-1} and H^-_{k-1}. We will show how to cover all the empty triangles of H_k with two vertices in H^+_{k-1} and one in H^-_{k-1} with $O(n \log n)$ triangulations. Label the elements of H^-_{k-1} from left to right as $p_0, \ldots, p_{\frac{n}{2}-1}$.

For each $0 \leq d \leq k-1$, proceed as follows: For every $p_j \in H^-_{k-1}$ join p_j to the endpoints of all the edges of H^+_{k-1} of depth d. This gives us a set $ID^+_{d,j}$ of interior-disjoint empty triangles. It is not hard to see that if $(d, j) \neq (d', j')$, then $ID^+_{d,j} \cap ID^+_{d',j'} = \emptyset$.

It is easy to see that the union of these sets covers all the empty triangles with two vertices in H^+_{k-1} and one in H^-_{k-1}. In a similar way, cover all the triangles with two vertices in H^-_{k-1}, and one in H^+_{k-1} with a family of sets $ID^-_{d,j}$.

Let ℓ_1 be the straight line connecting the leftmost point in H^+_{k-1} to the rightmost point in H^-_{k-1}, and ℓ_2 the straight line that connects the rightmost point in H^+_{k-1} with the leftmost point of H^-_{k-1}. Let q be a point slightly above the intersection point of ℓ_1 with ℓ_2.

It is clear that for each $ID^+_{d,j}$ there is exactly one empty triangle that contains q in its interior. This implies that q is contained in $\Omega(n \log n)$ empty triangles and thus $\Omega(n \log n)$ triangulations are necessary to cover all the empty triangles in H_k.

Now we show that $O(n \log n)$ of H_k triangulations are sufficient. Consider each set $ID^+_{d,j}$ and $ID^-_{d,j}$, and complete it to a triangulation. This gives us $O(n \log n)$ triangulations that cover all the triangles with vertices in both H^+_{k-1} and H^-_{k-1}.

Take a set of triangulations $\mathcal{T}^+_{k-1} = \{T^+_1, \ldots, T^+_m\}$ of H^+_{k-1} that covers all of its empty triangles. Since H_{k-1}^+ and H_{k-1}^- are isomorphic, we can find a set of triangulations $\mathcal{T}_{k-1}^- = \{T_1^-, \dots, T_m^-\}$ of H_{k-1}^- that covers all the empty triangles of H_{k-1}^- such that T_i^+ is isomorphic to T_i^-. For each i, we can find a triangulation T_i of H_k that contains T_i^+ and T_i^- as induced subgraphs.

Thus if $T(n)$ is the number of triangulations required to cover the empty triangles of H_k, the following recurrence holds for $n = 2^k$:

$$T(n) = T\left(\frac{n}{2}\right) + O(n \log n).$$

This solves to $T(n) = O(n \log n)$, and our result follows. □

We conclude this section with the following conjecture:

Conjecture 2. *At least $\Omega(n \log n)$ triangulations are needed to cover all the empty triangles of any point set with n points.*

4 A point in many edge-disjoint triangles

The problem of finding a point contained in many triangles of a point set was solved by Boros and Füredi [4], see also Bukh [6]. They proved:

Theorem 5. *For any set P of n points in general position, there is a point in the interior of the convex hull of P contained in $\frac{2}{9}\binom{n}{3} + O(n^2)$ triangles of P. The bound is tight.*

We now study a variant to this problem, in which we are interested in finding a point in many *edge-disjoint* triangles. We consider the following:

Problem 3. Let P be a set of points in the plane in general position, and $q \notin P$ a point of the plane. What is the largest number of edge-disjoint triangles of P such that q belongs to the interior of all of them?

We start by giving some preliminary results, and then we study Problem 3 for point sets in general position, and sets of vertices of regular polygons.

Given a point set P, and a point q not in P, let $\mathcal{T}(P, q)$ (or $\mathcal{T}(q)$ for short) be the set of triangles of P that contain q. We define the graph $G(P, q)$ whose vertex set is $\mathcal{T}(q)$ in which two triangles are adjacent if they share an edge; see Figure 4. We may assume that q does not belong to any line passing through two elements of P. We now prove:

Lemma 3. *The degree of every vertex of $G(P, q)$ is exactly $n - 3$.*

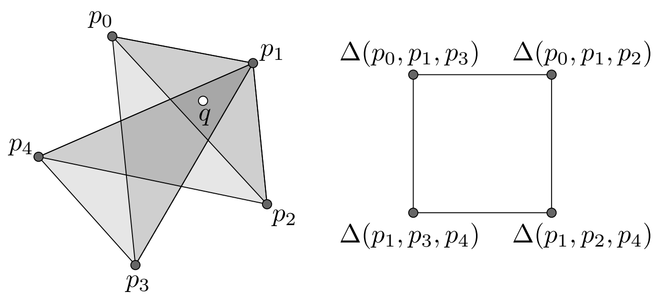

Figure 4: $G(P, q)$.

Proof. Let $\Delta(x, y, z)$ be a triangle that contains q in its interior. Let p be any point in $P \setminus \{x, y, z\}$. Then exactly one of the triangles $\Delta(x, y, p)$, $\Delta(x, p, z)$, or $\Delta(p, y, z)$ contains q; see Figure 5. That is, exactly one of $\Delta(x, y, p)$, $\Delta(x, p, z)$, or $\Delta(p, y, z)$ belongs to $\mathcal{T}(q)$. Our result follows. □

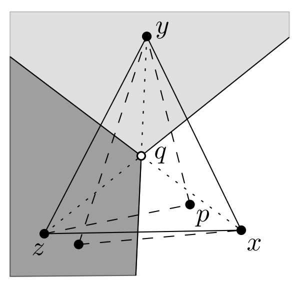

Figure 5:

Observe now that finding sets of edge-disjoint triangles that contain q is equivalent to finding independent sets in $G(P, q)$. Let $\tau(P, q)$ (or $\tau(q)$ for short) be the largest number of edge-disjoint triangles on P containing q. We now prove:

Lemma 4.

$$\frac{|\mathcal{T}(q)|}{n-2} \leq \tau(q) \leq \frac{3|\mathcal{T}(q)|}{n}.$$

Proof. It follows from Lemma 3 that the size of the largest independent set of $G(P, q)$ is at least $\frac{|\mathcal{T}(q)|}{n-2}$. To prove our upper bound, it is sufficient to observe that if a vertex of $G(P,q)$ is not in an independent set I of $G(P,q)$, then it is adjacent to at most three vertices in it, one per each of its edges. Hence by counting the number of edges connecting a vertex in I to another in $\mathcal{T}(q) \setminus I$, we obtain that:

$$(n-3)|I| \leq 3|\mathcal{T}(q) \setminus I|.$$

Our result follows. □

From Theorem 5 and Lemma 4 it is easy to see that in any set of n points in general position on the plane there is a point q such that

$$\frac{n^2}{27} + O(n) \approx \frac{\frac{2}{9}\binom{n}{3} + O(n^2)}{n-2} \leq \tau(q) \leq \frac{3 \cdot \frac{2}{9}\binom{n}{3} + O(n^2)}{n} \approx \frac{n^2}{9} + O(n).$$

Thus we have:

Corollary 2. *For any point set in general position on the plane there is a point q such that $\tau(q) \leq \frac{n^2}{9} + O(n)$.*

We now prove an even stronger result. We now prove:

Proposition 1. *Let P a set of n points in general position on the plane. Then for any point $q \notin P$ of the plane $\tau(q) \leq n^2/9$.*

Proof. Let $q \notin P$ be any point of the plane. If q is on a straight line passing through two elements of P, then by slightly moving it, q could be moved to a position in which it is contained in more edge-disjoint triangles. Thus assume that q is not on any straight line through two elements of P.

First we show the following lemma:

Lemma 5. *There exist three straight lines passing through q such that they partition P into six subsets $P_0, P_1, \ldots, P_5$ in counter-clockwise order around q, with $|P_0| = |P_2| = |P_4|$ (we allow the possibility that $P_i = \emptyset$ for some i).*

Proof. Let l_0 be a straight line passing through q such that one of the half-planes bounded by l_0, say the one on top of it, contains an even number of elements of P. Take other straight lines l_1 and l_2 passing through q, and define the subsets P_i of P, $0 \leq i \leq 5$, as shown in Figure 6 a), where $|P_0 \cup P_1 \cup P_2|$ is even. Let l^* be a straight line passing through q, equi-partitioning the elements of $P_0 \cup P_1 \cup P_2$.

Choose l_1 and l_2 such that initially $|P_0| = |P_2| = |P_3| = |P_5| = 0$. From their initial positions, rotate l_1 counter-clockwise and l_2 clockwise around q

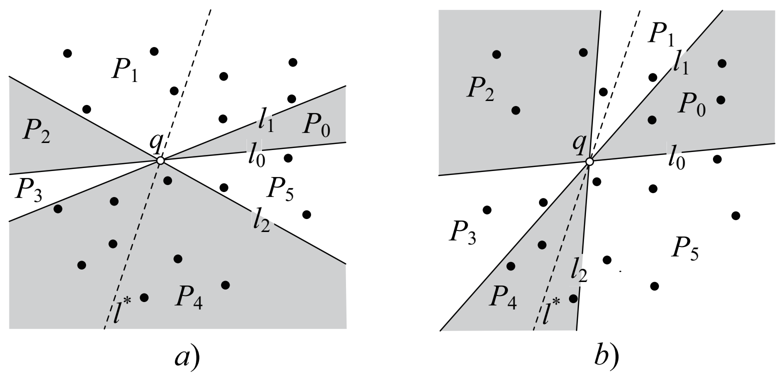

Figure 6: Partitions of P.

in such a way that P_0 and P_2 always contain the same number of elements, and until they both reach the position of l^* at the same time, and the boundary of P_4 always contains no more than one element of P.

Initially $|P_4| \geq 0 = |P_0|$. On the other hand, we have $|P_4| = 0 \leq |P_0|$ when l_1 and l_2 reach the position of l^*. Hence at some point while rotating l_1 and l_2, we have that $|P_0| = |P_2| = |P_4|$; see Figure 6 b). □

Let $P_0, P_1, \ldots, P_5$ be as in Lemma 5. Write $|P_i| = n_i$ for $0 \leq i \leq 5$ (we have $n_0 = n_2 = n_4$). We henceforth read indices modulo 6. Let $\mathcal{T}$ be a set of edge-disjoint triangles with vertices in P, containing q in its interior. For integers i, j, k, let $\mathcal{T}_{ijk}$ denote the set of elements of $\mathcal{T}$ such that it has one vertex in P_i, another in P_j and the other in P_k, and let $t_{ijk} = |\mathcal{T}_{ijk}|$; see Figure 7.

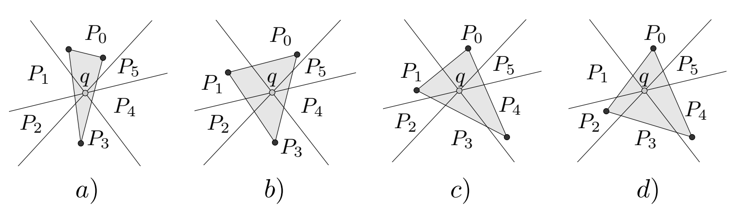

Figure 7: Triangles in the $\mathcal{T}_{ijk}$'s.

Then

$$\mathcal{T} = \left[\cup_{i=0}^{5}\mathcal{T}_{ii(i+3)}\right] \cup \left[\cup_{i=0}^{5}\mathcal{T}_{i(i+1)(i+3)}\right] \cup \left[\cup_{i=0}^{5}\mathcal{T}_{i(i+1)(i+4)}\right] \cup \left[\cup_{i=0}^{5}\mathcal{T}_{i(i+2)(i+4)}\right]$$
$$= \left[\cup_{i=0}^{5}\mathcal{T}_{ii(i+3)}\right] \cup \left[\cup_{i=0}^{5}\mathcal{T}_{i(i+2)(i+5)}\right] \cup \left[\cup_{i=0}^{5}\mathcal{T}_{i(i+2)(i+3)}\right] \cup \left[\cup_{i=0}^{5}\mathcal{T}_{i(i+2)(i+4)}\right].$$

For integers i, j, let E_{ij} denote the set of all segments connecting an element of P_i and another of P_j. Then for each integer i, $|E_{i(i+2)}| = n_i n_{i+2}$ and $\mathcal{T}_{i(i+2)(i+3)} \cup \mathcal{T}_{i(i+2)(i+4)} \cup \mathcal{T}_{i(i+2)(i+5)}$ is the set of elements of $\mathcal{T}$ which has a side belonging to $E_{i(i+2)}$. Hence we have

$$f(i) \equiv t_{i(i+2)(i+3)} + t_{i(i+2)(i+4)} + t_{i(i+2)(i+5)} \leq n_i n_{i+2} \tag{1}$$

for each i. Similarly, by considering the cardinality of $E_{i(i+3)}$, we obtain

$$\begin{aligned} g(i) \equiv 2t_{ii(i+3)} + t_{i(i+1)(i+3)} + t_{i(i+2)(i+3)} & \\ +2t_{i(i+3)(i+3)} + t_{i(i+3)(i+4)} + t_{i(i+3)(i+5)} & \leq n_i n_{i+3} \end{aligned} \tag{2}$$

for each i. By (1) and (2), we have

$$\sum_{i=0}^{5} f(i) + 2\sum_{i=0}^{2} g(i) \leq \sum_{i=0}^{5} n_i n_{i+2} + 2\sum_{i=0}^{2} n_i n_{i+3}. \tag{3}$$

Since $g(i) = (t_{i(i+2)(i+3)} + t_{j(j+2)(j+3)}) + (t_{j'(j'+2)(j'+5)} + t_{j''(j''+2)(j''+5)}) + 2(t_{ii(i+3)} + t_{jj(j+3)})$, where $j = i+3$, $j' = i+1$, $j'' = j'+3$,

$$\begin{aligned} \sum_{i=0}^{5} f(i) + 2\sum_{i=0}^{2} g(i) &= \sum_{i=0}^{5}(t_{i(i+2)(i+3)} + t_{i(i+2)(i+4)} + t_{i(i+2)(i+5)}) \\ &\quad +2\sum_{i=0}^{5}(t_{i(i+2)(i+3)} + t_{i(i+2)(i+5)}) + 4\sum_{i=0}^{5} t_{ii(i+3)} \\ &= 3|\mathcal{T}| + \sum_{i=0}^{5} t_{ii(i+3)} \geq 3|\mathcal{T}|. \end{aligned} \tag{4}$$

On the other hand, if we denote the right-hand side of (3) by S,

$$\begin{aligned} S &= (n_0n_2 + n_2n_4 + n_4n_0) + (n_1n_3 + n_3n_5 + n_5n_1) \\ &\quad +2(n_0n_3 + n_2n_5 + n_4n_1) \\ &= \frac{l^2}{3} + \frac{2lm}{3} + (n_1n_3 + n_3n_5 + n_5n_1), \end{aligned} \tag{5}$$

where $l = n_0+n_2+n_4$ (recall that $n_0 = n_2 = n_4$) and $m = n_1+n_3+n_5$. Since $n_1n_3+n_3n_5+n_5n_1 = [m^2-(n_1^2+n_3^2+n_5^2)]/2$ and since $n_1^2+n_3^2+n_5^2 \geq m^2/3$ with equality if and only if $n_1 = n_3 = n_5$, we have $n_1n_3 + n_3n_5 + n_5n_1 \leq m^2/3$. From this and (5), it follows that

$$S \leq \frac{l^2}{3} + \frac{2lm}{3} + \frac{m^2}{3} = \frac{(l+m)^2}{3} = \frac{n^2}{3}. \tag{6}$$

Now combining (3), (4) and (6), we obtain $|\mathcal{T}| \leq n^2/9$, as desired. □

To achieve the equality, it is necessary that $n_0 = n_2 = n_4$ and $n_1 = n_3 = n_5$ for some partition (Figure 8).

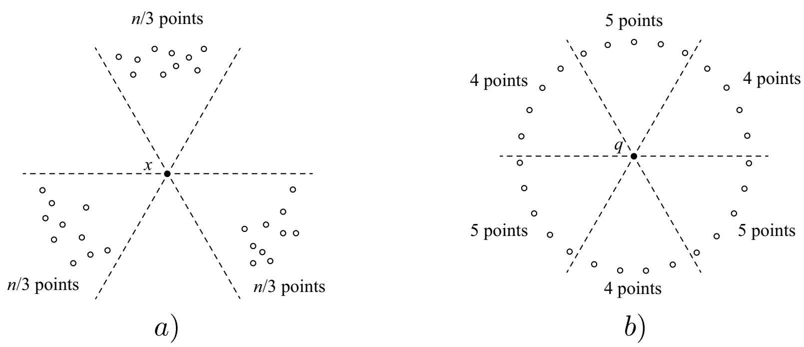

Figure 8: A vertex set of a regular 27-gon.

We now prove:

Proposition 2. *Let n be a positive integer and P a set of n points in general position on the plane. Then there exists a point q on the plane such that $\tau(q) \geq \frac{n^2}{12}$.*

Proof. We use the following lemma which was proved by Ceder [7] (see also [5]), and applied by Bukh [6] to obtain a lower bound of $\max_q |\mathcal{T}(q)|$ for given P:

Lemma 6. *There exist three straight lines such that they intersect at a point q and partition the plane into 6 open regions each of which contains at least $n/6 - 1$ elements of P.*

Let q be as in Lemma 6. We may assume that q is not on any straight line passing through two elements of P. Let $m = \lceil n/6 \rceil - 1$ and denote by $D_0, D_1, \ldots, D_5$ the six regions in counter-clockwise order around q. For each $0 \leq i \leq 5$, let P_i be a subset of $P \cap D_i$ with $|P_i| = m$; see Figure 9.

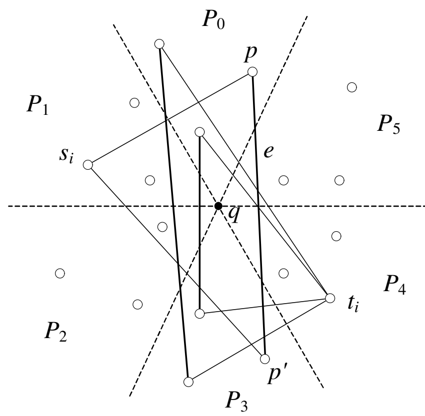

Figure 9: Matching M_i (bold lines) and triangles using edges of M_i.

Now consider the geometric complete bipartite graph with vertex set $P_0 \cup P_3$ and edge set $E = \{pp' \mid p \in P_0, p' \in P_3\}$. As a consequence of a well-known result in graph theory, E can be decomposed into m subsets M_i, $0 \leq i \leq m-1$, such that each M_i is a perfect matching, i.e., consisting of m independent edges. Let $P_1 = \{s_1, s_2, \ldots, s_m\}$ and $P_4 = \{t_1, t_2, \ldots, t_m\}$. For each i and each element $e = pp' \in M_i$, where $p \in P_0$ and $p' \in P_3$, let u_i denote either s_i or t_i according to whether $pp' \cap D_1 = \emptyset$ or $pp' \cap D_4 = \emptyset$. Then $\triangle(p, p', u_i)$ contains q in its interior. Observe that all of the m triangles in $\mathcal{T}_i = \{\triangle(p, p', u_i) \mid e = pp' \in M_i\}$ are edge-disjoint, and all of the m^2 triangles in $\mathcal{T}_{03} = \cup_{i=0}^{m} \mathcal{T}_i$ are edge-disjoint as well.

Define the sets $\mathcal{T}_{14}$ and $\mathcal{T}_{25}$ of triangles similarly (the elements of $\mathcal{T}_{14}$ are triangles with one vertex in P_1, another in P_4 and the other in $P_2 \cup P_5$, while the elements of $\mathcal{T}_{25}$ are triangles with one vertex in P_2, another in P_5 and the other in $P_3 \cup P_0$). It can be observed that all of the $3m^2 = n^2/12 - O(n)$ triangles in $\mathcal{T}_{03} \cup \mathcal{T}_{14} \cup \mathcal{T}_{25}$ are edge-disjoint. □

Thus by using Corollary 2, Proposition 1, and Proposition 2 we have:

Theorem 6. *In any point set in general position, there is a point q such that $\frac{n^2}{12} \leq \tau(q) \leq \frac{n^2}{9}$. Moreover, for any q, $\tau(q) \leq \frac{n^2}{9}$.*

4.1 Regular Polygons

By Theorem 6, any point in the interior of the convex hull of a point set is contained in at most $n^2/9$ edge-disjoint triangles of P. It is also easy to construct point sets for which that bound is tight; see Figure 8 a). In fact, the point sets in that figure can be chosen in convex position.

We now show that the bound in Theorem 6 is also achieved when P is the set of vertices of a regular polygon. We found proving this result to be a challenging problem. In what follows, we will assume that $n = 9m$, $m \geq 1$.

Let (a_i, b_i, c_i) be an ordered set of integers. We call (a_i, b_i, c_i) a *triangular triple* if it satisfies the following conditions:

a) a_i, b_i, and c_i are all different,

b) $a_i + b_i + c_i = n - 3$, and

c) $1 \leq a_i, b_i, c_i \leq \frac{n-3}{2}$.

Observe that for any vertex p_r of P, a triangular triple (a_i, b_i, c_i), defines a triangle $\Delta(p_r, p_{r+a_i+1}, p_{r+a_i+b_i+2})$ of P. Moreover, condition c) above ensures that $\Delta(p_r, p_{r+a_i+1}, p_{r+a_i+b_i+2})$ is acute, and hence it contains the center c of P. Note that since $a_i + b_i + c_i = n - 3$, $p_r = p_{r+a_i+b_i+c_i+3}$, addition taken mod n. Thus the edges of $\Delta(p_r, p_{r+a_i+1}, p_{r+a_i+b_i+2})$ skip a_i, b_i, and c_i vertices of P respectively; see Figure 10 a).

Let $S(a_i, b_i, c_i) = \{\Delta(p_r, p_{r+a_i+1}, p_{r+a_i+b_i+2}) : p_r \in P\}$. The set $S(a_i, b_i, c_i)$ can be seen as the set of triangles obtained by rotating $\Delta(p_0, p_{0+a_i+1}, p_{0+a_i+b_i+2})$ around the center of P; see Figure 10 b). The next observation will be useful:

Observation 2. *Let (a_i, b_i, c_i) and (a_j, b_j, c_j) be triangular triples of P such that $\{a_i, b_i, c_i\} \cap \{a_j, b_j, c_j\} = \emptyset$, $i \neq j$. Then all of the triangles in $S(a_i, b_i, c_i) \cup S(a_j, b_j, c_j)$ are edge-disjoint.*

Consider a set $C = \{(a_0, b_0, c_0), \ldots, (a_{k-1}, b_{k-1}, c_{k-1})\}$ of ordered triangular triples. We say that C is a *generating set* of triangular triples if the following condition holds:

$$\{a_i, b_i, c_i\} \cap \{a_j, b_j, c_j\} = \emptyset, \ i \neq j.$$

Note that $|S(a_i, b_i, c_i)| = n$, and thus

$$\bigcup_{(a_i,b_i,c_i) \in C} S(a_i, b_i, c_i)$$

contains nk edge disjoint triangles containing the center P. Our task is now that of finding a generating set of as many triangular triples as possible.

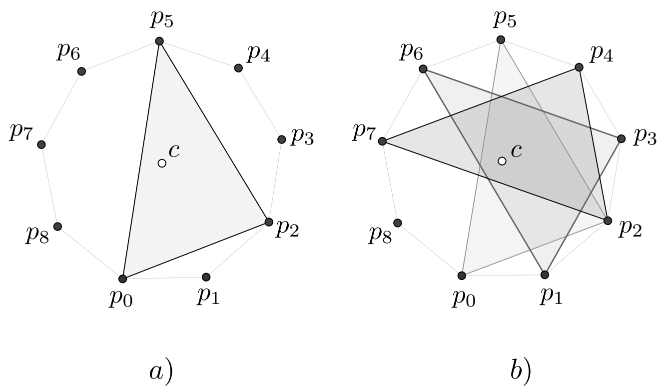

Figure 10: a) The triple $(1,2,3)$, and p_0 determine $\Delta(p_0,p_2,p_5)$. b) $S(1,2,3)$ is obtained by rotating $\Delta(p_0,p_2,p_5)$, obtaining a set of 9 edge-disjoint triangles.

Theorem 7. *Let P be the set of vertices of a regular polygon with $n=9m$ vertices, and let c be its center. Then if m is odd, then $|\tau(c)| \geq \frac{n^2}{9}$, and if m is even, then $|\tau(c)| \geq \frac{n^2}{9} - n$.*

Proof. The proof when m is odd proceeds by constructing a generating set C with $\frac{n}{9}$ triangular triples. Let $k = \frac{9m-3}{6}$ and $k' = k + 2m - 1$. Given $i \in \{0,1,\ldots,m-1\}$ we define the i-th ordered triple (a_i,b_i,c_i) as follows (see Figure 11):

$$\begin{aligned}
a_i &= k + i,\\
b_i &= \begin{cases} k' - 2i - 1 & if \quad i < \frac{m-1}{2},\\ k' - 2i + m - 1 & if \quad i \geq \frac{m-1}{2},\end{cases}\\
c_i &= \begin{cases} k' + i + 1 + \frac{m+1}{2} & if \quad i < \frac{m-1}{2},\\ k' + i + 1 - \frac{m-1}{2} & if \quad i \geq \frac{m-1}{2}.\end{cases}
\end{aligned}$$

We now prove that the triples (a_i,b_i,c_i) are triangular; that is,s $a_i + b_i + c_i = n - 3$. Since $b_i + c_i = 2k' - i + \frac{m+1}{2}$ for all i,

$$a_i + b_i + c_i = k + 2k' + \frac{m+1}{2} = 9m - 3.$$

It is easy to see that

$$\begin{aligned}
& & k \leq & \quad a_i \quad \leq k + m - 1,\\
k + m &= & k' - m + 1 \leq & \quad b_i \quad \leq k',\\
& & k' + 1 \leq & \quad c_i.
\end{aligned}$$

$(4,8,12)$	$(7,15,20)$	$(10,22,28)$	$(13,29,36)$	$(16,36,44)$
$(5,9,10)$	$(8,13,21)$	$(11,20,29)$	$(14,27,37)$	$(17,34,45)$
$(6,7,11)$	$(9,16,17)$	$(12,18,30)$	$(15,25,38)$	$(18,32,46)$
	$(10,14,18)$	$(13,23,24)$	$(16,23,39)$	$(19,30,47)$
	$(11,12,19)$	$(14,21,25)$	$(17,30,31)$	$(20,28,48)$
		$(15,19,26)$	$(18,28,32)$	$(21,37,38)$
		$(16,17,27)$	$(19,26,33)$	$(22,35,39)$
			$(20,24,34)$	$(23,33,40)$
			$(21,22,35)$	$(24,31,41)$
				$(25,29,42)$
				$(26,27,43)$

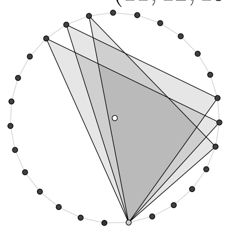

Figure 11: Triangular triples for $n = 27, 45, 63, 81$ and 99.

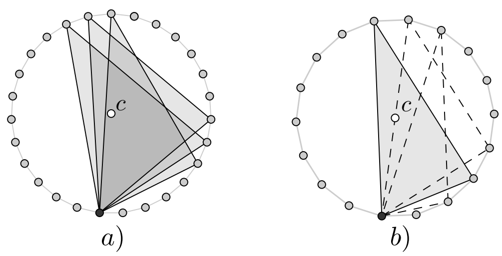

Figure 12: $a)$ Triangular triples (a_i, b_i, c_i) for $n = 9 \cdot 3 = 27$ and $b)$ triples $(a'_i, b'_i, c'_i) = (a_i - 3, b_i - 3, c_i - 3)$ for $n = 9 \cdot 2 = 18$.

Therefore $a_i < b_j < c_k$ for every i, j, k. Also, by construction it can be verified that $a_i \neq a_j$, $b_i \neq b_j$, and $c_i \neq c_j$ for every $i \neq j$.

$$\text{Thus the set} \bigcup_{(a_i,b_i,c_i)\in C} \{a_i, b_i, c_i\} \text{ contains no repeated elements.}$$

Finally, note that the maximum value that can be reached by c_i occurs when $i = \frac{m-3}{2}$, and thus:

$$c_i \leq k' + 1 + \frac{m-3}{2} + \frac{m+1}{2} = k' + m = \frac{9m-3}{2}.$$

Therefore C is a generating set of triangular triples. Thus c is contained in at least $\frac{n^2}{9}$ edge-disjoint triangles.

The proof when m is even proceeds by also constructing a set of m triples. We use the set of triples just constructed for $m+1$ and modify it as follows: Suppose that the set of $m+1$ triples is $\{(a_0, b_0, c_0), \ldots, (a_m, b_m, c_m)\}$.

Let $a'_i = a_i - 3$, $b'_i = b_i - 3$ and $c'_i = c_i - 3$ and consider $C' = \{(a'_i, b'_i, c'_i) \mid 0 \leq i \leq m\}$. C' induces a set of triangles in P. Nevertheless $2n$ triangles do not contain the point c in their interior; see Figure 12. Therefore this construction guarantees that c is contained in at least $(m+1)n - 2n = \frac{n^2}{9} - n$ edge-disjoint triangles. □

5 A point in many edge-disjoint empty triangles

We conclude our paper by briefly studying the problem of the existence of a point contained in many edge-disjoint empty triangles of a point set. We point out that if we are interested only in empty triangles containing a point, it is easy to see that for any point set P, there is always a point q contained in a linear number of (not necessarily edge-disjoint) empty triangles. This follows directly from the following facts:

1. Any point set P with n elements always determines at least a quadratic number of empty triangles [2, 16].

2. We can always choose $2n - c - 2$ points in the plane such that any empty triangle of P contains one of them, where c is the number of vertices of the convex hull of P; see [8, 16].

We now prove:

Theorem 8. *There are point sets P such that every $q \notin P$ is contained in at most a linear number of empty edge-disjoint triangles of P.*

Proof. Let H_k, H^+_{k-1} and H^-_{k-1} be as defined in Section 2. Consider any set T^+_k (respectively T^-_k) of empty edge-disjoint triangles such that each of them has two vertices in H^+_{k-1} (respectively H^-_{k-1}) and the other in H^-_{k-1} (respectively H^+_{k-1}). Let $t \in T^+_k$. Then the edge of t with both endpoints in H^+_{k-1} is an edge of H^+_{k-1} visible from below. Since the triangles in T^+_k are edge-disjoint, the number of elements of T^+_k is at most the number of edges of H^+_{k-1} visible from below, which is a linear function in n. Thus $|T^+_k| \in O(n)$. Similarly we can prove that $|T^-_k| \in O(n)$.

Consider a point $q \in \mathrm{CH}(H_k) \setminus \mathrm{CH}(H^+_{k-1}) \cup \mathrm{CH}(H^-_{k-1})$. Clearly any empty triangle containing q belongs to some $T^+_k \cup T^-_k$, and thus it belongs to at most a linear number of edge-disjoint triangles of H_k.

Suppose next that $q \in \mathrm{CH}(H_{k-1}^+) \cup \mathrm{CH}(H_{k-1}^-)$. Suppose without loss of generality that $q \in \mathrm{CH}(H_{k-1}^+)$, and that q belongs to a set S of edge-disjoint triangles of H_k. S may contain some triangles with vertices in both of H_{k-1}^+ and H_{k-1}^-. There are at most a linear number of such triangles. The remaining elements in S have all of their vertices in H_{k-1}^+. Thus the number of edge-disjoint triangles containing q satisfy

$$T(n) \leq T\left(\frac{n}{2}\right) + \Theta(n),$$

and thus q belongs to at most a linear number of edge-disjoint triangles.

The first part of our result follows. To show that our bound is tight, let q be as in the proof of Theorem 4. It is easy to see that q belongs to a linear number of triangles with vertices in both of H_k^+ and H_k^-, and our result follows. $\square$

We conclude with the following:

Conjecture 3. *Let P be a set of n points in general position on the plane. Then there is always a point $q \notin P$ on the plane such that it is contained in at least $\log n$ edge-disjoint triangles of P.*

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