On the number of edges in geometric graphs without empty triangles

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Abstract

In this paper we study the extremal type problem arising from the question: What is the maximum number $ET(S)$ of edges that a geometric graph G on a planar point set S can have such that it does not contain empty triangles? We prove:

$$\binom{n}{2} - O(n \log n) \leq ET(n) \leq \binom{n}{2} - \left(n - 2 + \left\lfloor \frac{n}{8} \right\rfloor\right).$$

keywords: Geometric graphs Empty triangles Combinatorial Geometry Extremal Problem

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1 Introduction

Let S be a set of n points on the plane in general position. A geometric graph G on S is a graph whose vertices are the points of S and whose edges are line segments joining pairs of points of S. Let p, q, r be three points on the plane. We say that the triangle $T(p, q, r)$ with vertex set $\{p, q, r\}$ is a triangle of G if the vertices and edges of $T(p, q, r)$ are vertices and edges of G.

We say that a triangle $T(p, q, r)$ of G is *empty* if it contains no points of S in its interior. In this paper we study the extremal type problem arising from the following question:

Problem 1. What is the maximum number of edges that a geometric graph on a planar point set S can have such that it does not contain empty triangles?

For a given point set S, let $ET(S)$ be the maximum integer such that there is a geometric graph on S with $ET(S)$ edges and containing no empty triangles. Additionally, let $ET(n)$ be the largest possible value of $ET(S)$ taken over all the point sets S of n points.

For example, it is easy to see that for any set S of four points, there is a geometric graph on S with exactly four edges containing no empty triangles. Moreover, any geometric graph on S with more than four edges always contains an empty triangle; see Fig. 1. Thus $ET(4) = 4$.

Figure 1: Triangle-free geometric graphs on four points.

The main result of this paper is the following:

$$\binom{n}{2} - O(n \log n) \leq ET(n) \leq \binom{n}{2} - \left(n - 2 + \left\lfloor \frac{n}{8} \right\rfloor\right).$$

Several results related to our problem have been studied in the past. For example, a well known result in graph theory is Turán's Theorem [8] which states the following: *The maximum number of edges in a graph with n vertices containing no subgraph isomorphic to* K_{r+1} *is at most* $(1 - \frac{1}{r})\frac{n^2}{2}$. Notice that for $r = 2$, Turán's Theorem tells us that any graph that contains no triangles contains at most $\frac{n^2}{4}$ edges. Thus if the elements of S are the vertices of a convex polygon, then $ET(S) = \frac{n^2}{4}$. This follows since in this configuration any triangle with vertices in S is empty.

In regard to geometric graphs, Nara, Sakai and Urrutia [6] give sharp bounds for the next problem: *What is the largest number of edges that a geometric graph with n vertices may have in such a way that it does not contain a convex r-gon?* We observe that for $r = 3$, this problem is not the problem studied here, as in this paper we deal with empty triangles.

A k-hole of S is a convex k-gon whose vertices are points of S containing no point of S in its interior. Erdős [2] asked about the existence of k-holes in planar point sets. Horton [5] proved that for $k \geq 7$ there are point sets containing no k-holes. Nicolás [7] proved that any point set with sufficiently many points contains a 6-hole. A second proof of the same result was later obtained by Gerken [3]. It follows trivially that $\binom{n}{2}$ is the maximum number of edges that a geometric graph can have in such a way that it contains no empty convex k-gons, $k \geq 7$.

In Sections 2 and 3 respectively we give lower and upper bounds for $ET(n)$. In Section 4 we present some conclusions and propose some open problems.

2 Lower Bound

In this section we show that $\binom{n}{2} - O(n \log n) \leq ET(n)$. *Horton sets* will be key elements for achieving this result. Horton [5] recursively constructed a set H_k of size 2^k, for any positive integer k such that H_k has no 7-holes. The construction is as follows:

(a) $H_1 = \{(0,0), (1,0)\}$.

(b) H_k consists of two subsets of points H^-_{k-1} and H^+_{k-1} obtained from H_{k-1} as follows: If $p = (i,j) \in H_{k-1}$, then $p' = (2i, j) \in H^-_{k-1}$ and $p'' = (2i+1, j+d_k) \in H^+_{k-1}$. The value d_k is chosen large enough such that any line ℓ passing through two points of H^+_{k-1} leaves all the points of H^-_{k-1} below it.

The next observations follow from the definition of H_k.

Observation 1. The points $p = (i,j)$ of H_k with i even belong to H^-_{k-1}, and those with i odd belong to H^+_{k-1}.

Let p and q be two points in the plane, and consider the vertical lines ℓ_p and ℓ_q passing trough them. Consider the vertical strip $ST_{p,q}$ bounded by ℓ_p and ℓ_q. Given a point set S, and $p, q \in S$. We say that the segment $\overline{pq}$ has no points below it (respectively above it) if there is no element of S in $ST_{p,q}$ below (resp. above) the segment $\overline{pq}$.

Let $B(H_k)$ be the set of line segments joining pairs of points of H_k such that there are no points of H_k below them. See Figure 2.

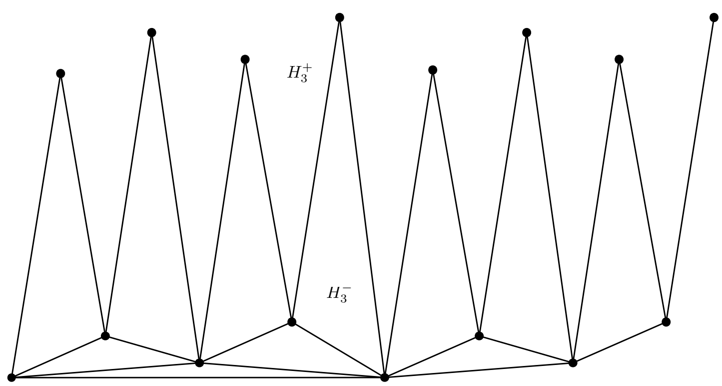

Figure 2: The set line segments of $B(H_4)$.

Observation 2. If $p, q \in H_{k-1}^+$, then there is at least one point of H_{k-1}^- below the line segment $\overline{pq}$. If $p, q \in H_{k-1}^-$ then there is a point of H_{k-1}^+ above $\overline{pq}$.

A proof of the following lemma was also given by Bárány and Valtr in [1].

Lemma 1. $|B(H_k)| = 2^{k+1} - (k+2)$.

Proof. The proof is by induction on k. It is clear that for $k = 1$ our result is true. Let $H_k = H_{k-1}^- \cup H_{k-1}^+$ as above, and let $p, q \in H_k$ such that the segment $\overline{pq}$ has no point of H_k below it. The following cases arise. Suppose first that p and q belong to H_{k-1}^-. By induction there are $2^k - (k+1)$ such pairs. Clearly by Observation 2, p and q cannot both belong to H_{k-1}^+. Thus the only remaining case is that one of them is in H_{k-1}^- and the other is in H_{k-1}^+. By Observation 1, it is easy to see that if $p = (i, j)$, then $q = (i \pm 1, r)$ for some r, and thus there are exactly $2^k - 1$ such pairs. Hence

$$|B(H_k)| = 2^k - (k+1) + 2^k - 1 = 2^{k+1} - (k+2)$$

and our result follows. □ □

We remark that the only segments in $B(H_k) \setminus B(H_{k-1}^-)$ are those joining pairs of points such that one of them belongs to H_{k-1}^+ and the other to H_{k-1}^-. Thus we have:

Observation 3. $|B(H_k)| - |B(H_{k-1}^-)| = 2^k - 1$.

It is easy to see that the number of segments joining pairs of points in H_k having no elements of H_k *above* them is also $2^{k+1} - (k+2)$.

Hence Lemma 1 states that the number of segments joining pairs of elements of H_k such that there is no element of H_k below them, (namely $2 \times 2^k - (k+2)$), is linear with respect to 2^k, the cardinality of H_k.

We will use Lemma 1 to prove the following theorem:

Theorem 1. For every $n = 2^k$, $k \geq 1$, there is a point set (namely H_k) such that there is a geometric graph on H_k with $\binom{n}{2} - O(n \log n)$ edges with no empty triangles.

Proof. Let G_k be the geometric graph on H_k obtained as follows: G_1 contains the edge joining $(0,0)$ to $(1,0)$. Suppose now that we have constructed G_{k-1} on H_{k-1}. G_k is obtained as follows:

1. If the edge pq is in G_{k-1}, then the edges $p'q'$ and $p''q''$ are in G_k, where p', q', p'' and q'' are as in item (b) in the definition of H_k.

2. Add to G_k all the edges joining pairs of points p, q such that $p \in H_{k-1}^+$, and $q \in H_{k-1}^-$.

3. Finally, remove from G_k all the edges joining pairs of vertices in H_{k-1}^+ that have no elements of H_{k-1}^+ below them, plus all the edges joining pairs of vertices in H_{k-1}^- that have no elements of H_{k-1}^- *above* them; see Figure 3.

We now show, by induction on k, that G_k contains no empty triangles. Our result is true for $k = 1$. Suppose that G_{k-1} contains no empty triangles. By induction, if G_k contains an empty triangle T, it must have two vertices in H_{k-1}^+ and one vertex in H_{k-1}^-, or vice versa.

Suppose that T has two vertices $p, q \in H_{k-1}^+$. Then it is easy to see that the line segment $\overline{pq}$ has no point of H_{k-1}^+ below it. But all such edges were removed from G_k in the third step of the construction above. In a similar way we can prove that T cannot contain two vertices in H_{k-1}^-.

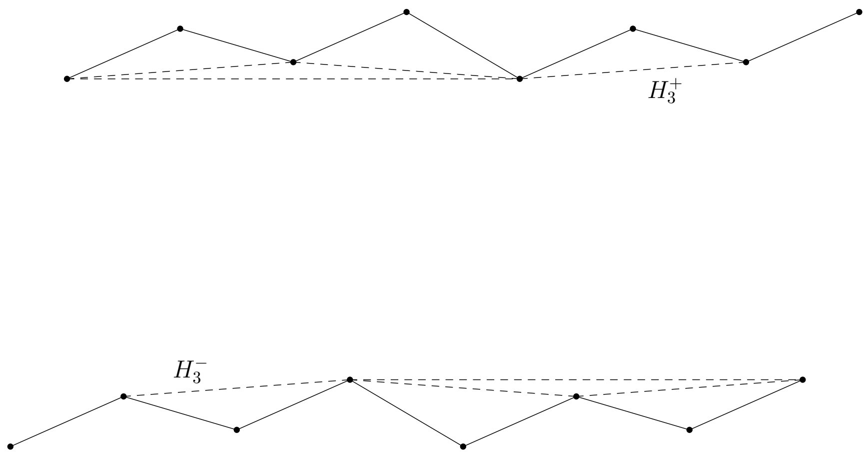

Figure 3: The solid and dashed edges are removed to construct G_4. As it turns out, the dashed edges were already removed when G_3 was constructed.

To show that G_k contains $\binom{n}{2} - O(n \log n)$ edges, we notice that by Lemma 1, in the third step of our construction of G_k we remove at most $2(2^k - (k+1))$ edges from the complete graph on H_k. Since the number of vertices in H_k is $n = 2^k$, the number of edges that we remove from the complete graph on H_k is bounded by the recursion

$$g(n) \leq 2g\left(\frac{n}{2}\right) + 2n,$$

which implies that $g(n)$ is bounded above by $O(n \log n)$. Our result follows.

□ □

As it turns out, by using Observation 3, and the fact noted in the caption of Figure 3, we can in fact count exactly the number of edges removed from the complete graph on H_k, which is:

$$n \log_2 n - 2n + 2 = 2^k(k-2) + 2\,.$$

The details are left to the reader.

Thus by Theorem 1, for $n = 2^k$, $\binom{n}{2} - O(n \log n) \leq ET(n)$.

Since $ET(n)$ is monotonically increasing, we now have that for any $2^{k-1} < m < 2^k$ $\binom{m}{2} - O(m \log m) \leq ET(n)$ (simply remove $2^k - m$ points from H_k). Thus for every n, and by the definition of $ET(n)$ we have:

Theorem 2. $\binom{n}{2} - O(n \log n) \leq ET(n)$.

We observe that Theorem 1 can be generalized to empty quadrilaterals, pentagons and hexagons of H_k. That is G_k, contains no empty quadrilaterals, pentagons, and hexagons. This follows from the observation that any empty polygon with vertices in H_k having vertices in H_{k-1}^+ and in H_{k-1}^-, has two adjacent vertices p and q, both in H_{k-1}^+ or in H_{k-1}^-. In the first case, $\overline{pq}$ has no elements of H_{k-1}^+ below it, in the second case $\overline{pq}$ has no elements of H_{k-1}^- above it. For empty convex polygons with more than seven vertices, we remove no edges from the complete geometric graph on H_k, as H_k contains no such empty convex polygons.

3 Upper Bound

In this section we provide an upper bound for $ET(n)$. Observe that for $n \geq 3$, $ET(n) < \binom{n}{2}$, as the complete graph K_n on any point set with n elements in general position has at least one empty triangle. A non-trivial upper bound is given by the following theorem.

Theorem 3. $ET(n) \leq \binom{n}{2} - \left(n - 2 + \left\lfloor \frac{n}{8} \right\rfloor\right)$.

Proof. Let S be any set of n points in general position, and K_n the complete geometric graph with vertex set S. Let F be a set of f edges of K_n such that $G = K_n \setminus F$ contains no empty triangles, and for any subset F' of edges of K_n with $|F'| < f$, $K_n \setminus F'$ contains empty triangles. We call the edges in F *forbidden edges.* Clearly $ET(S) = \binom{n}{2} - f$. The following is easy to prove:

Observation 4. There is at least one point $u \in S$ lying on the boundary of the convex hull of S such that u is incident to a forbidden edge $e \in F$.

To start, assign e to the vertex u. Then discard u and e and recursively apply Observation 4 to the geometric graph $G \setminus \{u\}$ until only two points are left. At the end of the process, we have associated $n-2$ forbidden edges to $n-2$ different points in S; moreover these edges form a forest $\mathcal{B}$ of forbidden edges.

Next we show how to find $\left\lfloor \frac{n}{8} \right\rfloor$ additional forbidden edges. Let p be the point in S with the lowest y-coordinate, and let $p_1, \ldots, p_{n-1}$ be the elements of $S \setminus \{p\}$, sorted radially from p in the clockwise order. Set $S_i = \{p_{8i+1}, \ldots, p_{8i+9}\} \cup \{p\}$, for $0 \leq i < \lfloor \frac{n}{8} \rfloor$. Observe that $|S_i| = 10$; see Figure 4.

It is known that every set of ten points on the plane contains an empty convex pentagon [4]. Let $P_i \subset S_i$ such that its elements are the vertices of a convex empty pentagon.

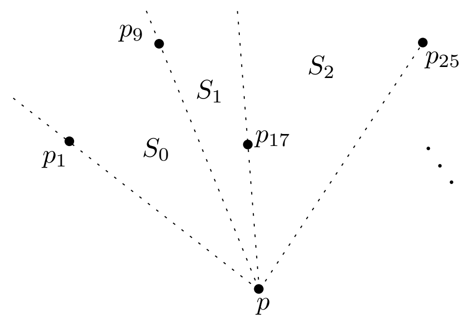

Figure 4: Splitting S into subsets of ten points.

Let Q_5 be any set of five points in convex position, and K_5 the complete geometric graph on Q_5. It is not hard to see that we need to remove at least four edges from K_5 to obtain a geometric graph with no empty triangles. In fact, there are only two such subsets of edges that are combinatorially different. Moreover each of these configurations of removed edges contains a cycle which is, in fact, a triangle; see Figure 5. Additionally, any geometric graph on Q_5 with five or more edges contains a cycle. Therefore each P_i should contain at least one cycle C^i of forbidden edges, one of which is not in $\mathcal{B}$, as $\mathcal{B}$ is a forest. Our result would follow if all the cycles C^i were edge-disjoint; see Figure 6.

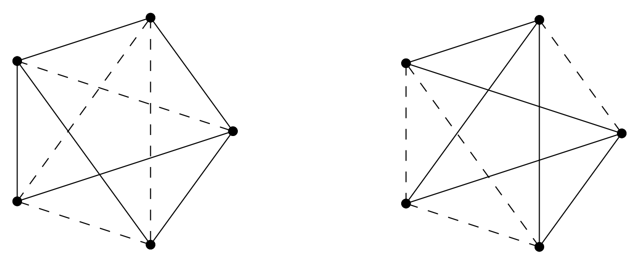

Figure 5: K_5 without dashed edges contains no empty triangle.

Let i and j be two integers with $1 \leq i < j \leq \lfloor \frac{n}{8} \rfloor$ and such that for any r, $i \leq r < j$, C^r and C^{r+1} share an edge. It is easy to see that C^r and C^{r+1} share two vertices, one of which is p. Thus, the graph $C^i \cup \cdots \cup C^j$ contains $(|C^i| - 2) + \cdots + (|C^j| - 2) + 2$ vertices, and $(|C^i| - 1) + \cdots + (|C^j| - 1) + 1$ edges. To destroy all the cycles in $C^i \cup \cdots \cup C^j$, we need to remove at least $j - i + 1$ edges. Our result follows now easily, and thus

$$f \geq n - 2 + \left\lfloor \frac{n}{8} \right\rfloor .$$

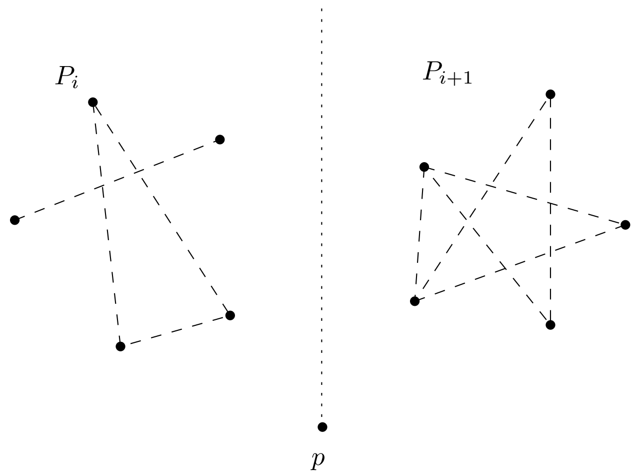

Figure 6: Each cycle of forbidden edges contains at least one edge that has not been counted before.

Hence

$$ET(S) = \binom{n}{2} - f \leq \binom{n}{2} - \left(n - 2 + \left\lfloor \frac{n}{8} \right\rfloor\right),$$

and

$$ET(n) \leq \binom{n}{2} - \left(n - 2 + \left\lfloor \frac{n}{8} \right\rfloor\right).$$

□ □

Summarizing, we have:

Theorem 4.

$$\binom{n}{2} - O(n \log n) \leq ET(n) \leq \binom{n}{2} - \left(n - 2 + \left\lfloor \frac{n}{8} \right\rfloor\right).$$

4 Conclusions and open problems

We obtained lower and upper bounds for the largest number of edges, $ET(n)$, that a geometric graph on any n-point set in general position can have such that it does not contain empty triangles. We conjecture that the lower bound

$$\binom{n}{2} - O(n \log n) \leq ET(n)$$

is tight. An open and interesting question is that of improving on the upper bound of Theorem 4. The problem of determining $ET(n)$ can be seen as a geometric version of Turán's Theorem [8], except that we focus on empty triangles rather than on complete graphs of order r.

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