On the number of non-crossing rays configurations

Alfredo García*, Ferran Hurtado†, Javier Tejel*, and J. Urrutia‡

Abstract

Given a set S of n points in R^2, we study the number of different ways of drawing n non-crossing rays, each one emanating from a point of S. If we denote by $r(S)$ this number, and call $r(n) = \max_{|S|=n} r(S)$, we prove that $\lim r(n)^{1/n} = 4$. We also consider the related problem of counting in how many different ways the points in S can be connected to a given curve by means of pairwise non-crossing segments.

1 Introduction

Let S be a set of n points in the plane, labelled $\{p_1, \ldots, p_n\}$, or simply $\{1, \ldots, n\}$ when this can cause no ambiguity, with no three of them on a line. Consider for each point p_i a ray r_i with apex p_i in such a way that the rays r_i are pairwise non-crossing. Then any circle enclosing S is cut by the rays in a cyclic order $r_{\pi(1)}, \ldots, r_{\pi(n)}$, where π is a permutation of $1, \ldots, n$ in which we can take $\pi(1) = 1$. We are interested in studying the number $r(S)$ of different cyclic permutations that can be obtained in this way from the point set S (see Figure 1). Counting exactly this number appears to be as a quite difficult problem even for very regular configurations of points, therefore we focus on obtaining tight bounds for $r(S)$, and looking for configurations of points achieving the extremal value, that we denote by

$$r(n) = \max_{|S|=n} r(S).$$

To the best of our knowledge this natural problem has not been previously studied, in spite of the fact that counting configurations consisting of non-crossing segments connecting points in the plane, such as polygons, trees, matchings or triangulations, has been a very active area of research for several years [1, 2, 3, 4, 6, 8, 9, 13, 14, 15, 16] that keeps attracting substantial attention.

A related problem consists of, given a point set $S = \{p_1, \ldots, p_n\}$ and a simple curve γ, induce a permutation on γ by connecting the points in S with γ by means of pairwise non-crossing line segments; when γ is a closed Jordan curve, we would consider the cyclic permutations induced on γ (see Figure 2). We call this problem *the γ-matching permutation problem.* One may think of it as a special case of the non-crossing rays problem in which we stop the rays as they hit the curve.

The paper is organized as follows. We consider the γ-matching permutation problem in Section 2 for the case in which γ is a line and all the points of S lie in one of the halfplanes defined by γ, and in Section 4 for points sets S enclosed by a convex curve γ. The results from Section 2 are used in Section 3 for studying the non-crossing rays problem, which we visit again in Section 5 for point sets evenly distributed on a circumference.

*Departamento de Métodos Estadísticos, Facultad de Ciencias, Universidad de Zaragoza. Pedro Cerbuna, 12, 50009-Zaragoza, Spain. {olaverri,jtejel}@unizar.es. Partially supported by projects MEC MTM2006-01267 and Aragón Govermment E58-DGA.

†Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya. Jordi Girona, 1-3, 08034-Barcelona, Spain. Ferran.Hurtado@upc.edu. Partially supported by projects MEC MTM2006-01267 and DURSI 2005SGR00692

‡Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la investigación cientifica, Circuito Exterior, Ciudad Universitaria, Coyoacán 04510, México, D.F. México. urrutia@matem.unam.mx. Supported by CONACYT of Mexico, Proyecto SEP-2004-Co1-45876, and PAPIIT (UNAM), Proyecto IN110802.

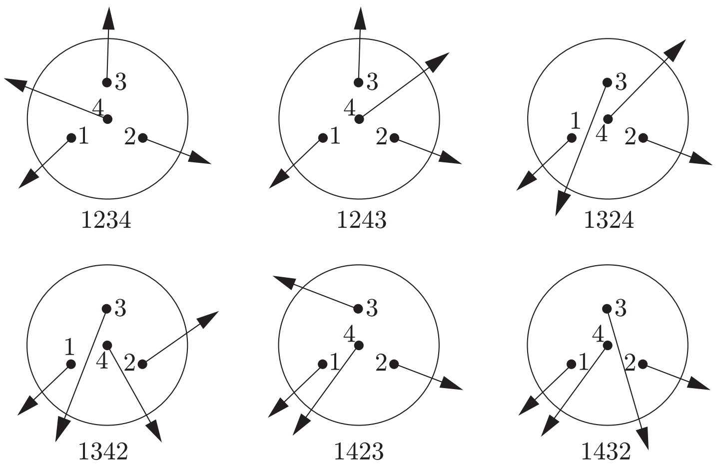

Figure 1: Cyclic permutations induced by non-crossing rays.

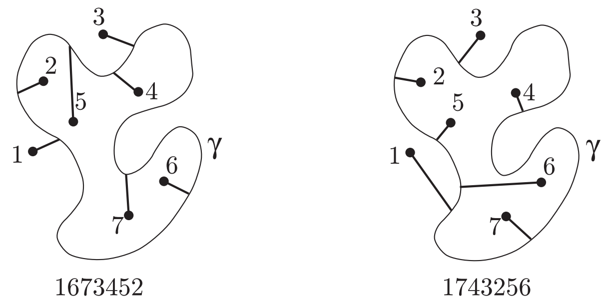

Figure 2: Two permutations induced by a point set on a curve.

2 The γ-matching permutation problem for halfplanes

Let l be a line and $S = \{p_1, \ldots, p_n\}$ be a set of points lying on a halfplane H limited by l. Without loss of generality we can assume that l is a horizontal line, H its upper halfplane, and that the points p_i are sorted in decreasing order of their ordinates. An *l-matching permutation* (or *matching permutation* for short) is defined as follows: Each point p_i is joined with a segment to a distinct point on the line l in such a way that the segments are pairwise non-crossing (see Figure 3). By traversing l from left to right we find first a point matched with some $p_{i_1} \in S$, then a point matched with $p_{i_2} \in S$, and so on. The sequence of indices $i_1, i_2, \ldots, i_n$ is the matching permutation induced by the set of segments. We say that a permutation of the numbers $1, 2, \ldots, n$ is a *feasible permutation* when it can be obtained as a matching permutation. The number of feasible permutations will be denoted by $r^l(S)$ and the extremal value that we study here is

$$r^l(n) = \max_{|S|=n} \{r^l(S)\}.$$

Notice that $r^l(1) = 1$. For convenience, we define as well $r^l(0)$ to be 1.

Notice that if two points have equal y-coordinates they must be connected to l in the order of their abscissae; in particular only one l-matching is possible if all the points of S lie on a line parallel to l. In the next statement we exclude these situations to avoid trivial lower bounds, yet observe that the upper bound applies to all configurations. We denote by C_n the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n} \in \Theta(4^n n^{-\frac{3}{2}})$, [18].

Lemma 2.1. *Let S any set of n points in the plane such that no two of them have equal y-coordinate. Then*

$$2^n - n \leq r^l(S) \leq C_n.$$

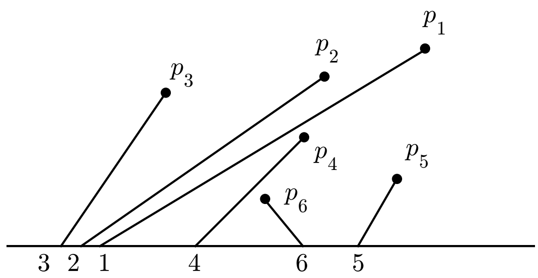

Figure 3: Feasible permutation 321465.

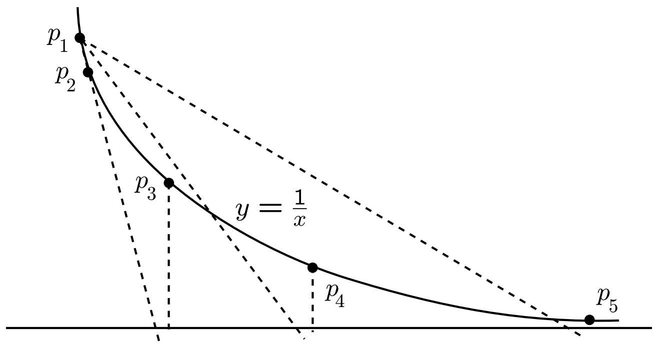

Figure 4: A configuration that achieves the maximum number of matchings.

Theorem 2.2. *The maximum of the values $r^l(S)$ when $|S| = n$ is*

$$r^l(n) = C_n.$$

The bound $r^l(n) \leq C_n$ is implied by Lemma 2.1; in order to prove equality, a specific set S with $r^l(S) = C_n$ is constructed (see Figure 4); its description is omitted from this extended abstract.

The point set construction used in the proof of the previous result has two additional properties: On one hand, any feasible permutation can be realized using matching segments that if extended downwards become pairwise noncrossing. On the other hand all these rays may be taken inside a cone of directions forming an angle as small as desired with the $+x$-axis.

Combining the first property with Theorem 2.2, we obtain the following result:

Corollary 2.3. *Let $r(S)$ be the number of different cyclic permutations that can be obtained shooting a set of pairwise noncrossing rays from a given set S of n points. Then*

$$r(n) = \max_{|S|=n} \{r(S)\} \geq C_n.$$

The lower bound given in Lemma 2.1 is not tight for $n \geq 5$, because in that proof only permutations corresponding to some sets of rays are counted. However, the following proposition shows that in fact the bound is quite tight. The proof requires the construction of a special point set and is omitted from this abstract.

Proposition 2.4. *There are point sets S of n points such that $r^l(S) \leq (8/5)2^n$.*

3 Counting configurations of non-crossing rays

We focus here in this section in the first problem mentioned in this paper: S is a set of n points in the plane, labelled $\{p_1, \ldots, p_n\}$, no three on a line, and from each point p_i we shoot a ray r_i to infinity, in such a way that the rays r_i are pairwise non-crossing. We are interested in bounding the the number $r(S)$ of different cyclic permutations that such sets of rays can generate and in the extremal value. $r(n) = \max_{|S|=n} r(S)$.

First of all, observe that any given configuration of non-crossing rays can be transformed into what we call a *canonic* configuration that induces the same feasible permutation, by clockwise rotating all the rays as much as possible (see Figure 5). In this way the ray from a point p_i is rotated until another point p_j is found, or until it is parallel to another ray in the direction of a line $p_j p_k$. Notice that in a canonic position the rays can only have one of the $\binom{n}{2}$ directions the points define.

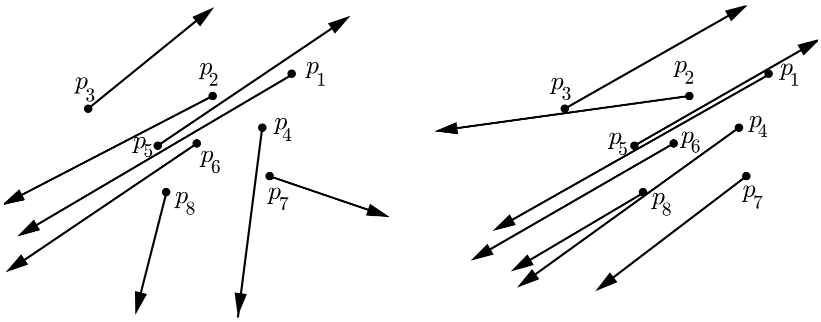

Figure 5: The canonic configuration of the permutation 12357486.

The technique in the proof of our next result permutation uses a distinction between feasible permutations that are called *separable* and those that are *nonseparable*. The first type are the permutations such that can be realized by some configuration of rays in such a way that it is possible to draw some line l in the plane that doesn't cross any ray.

Proposition 3.1. *There are polynomials $p(n)$ and $p'(n)$ such that $deg(p) \leq 9$, $deg(p') \leq 2$, and for any set S with n points we have*

$$p'(n)2^n \leq r(S) \leq p(n)C_n.$$

We believe the lower bound in the preceding proposition is quite tight; in fact we think possible to find sets S with n points such that $r(S) < q(n)2^n$ where $q(n)$ is some fixed polynomial, but the proof remains elusive to us. As for the upper bound, we know from Corollary 2.3 that it is tight up to polynomial terms, hence we can state the following result:

Theorem 3.2. *Let $r(S)$ be the number of different cyclic permutations that can be obtained shooting sets of pairwise noncrossing rays from a given set S of n points, and let us consider the extremal value $r(n) = \max_{|S|=n}$. Then*

$$\lim r(n)^{1/n} = 4.$$

4 The γ-matching permutation problem for convex regions

Let C be a closed Jordan curve bounding a convex region R^C, and let $S = \{p_1, \ldots, p_n\}$ be a set of points inside R^C. A *C-matching permutation* (or *matching permutation* for short) is defined as follows: Each point p_i is joined with a segment to a distinct point on the curve C in such a way that the segments are pairwise non-crossing (see Figure 6). By walking on C, say clockwise, we find first a point matched with some $p_{i_1} \in S$, then a point matched with $p_{i_2} \in S$, and so on. The cyclic sequence of indices $i_1, i_2, \ldots, i_n$ is the matching permutation induced by the set of segments. We say that a cyclic permutation of the numbers $1, 2, \ldots, n$ is a *feasible permutation* when it can be obtained as a matching permutation. The number of feasible permutations will be denoted by $r^C(S)$ and we give in this section bounds for this number.

Observe that if we take an increasing sequence of nested convex regions, $R^C = R^{C_0} \subset R^{C_1} \subset R^{C_2} \ldots$ then $r^{C_0}(S) \geq r^{C_1}(S) \geq r^{C_2}(S) \geq \ldots$. In addition, if C_i encloses all the intersection points between

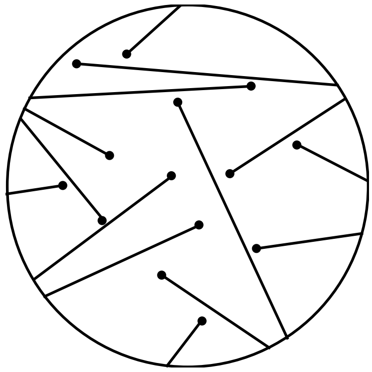

Figure 6: The γ-matching permutation problem inside a convex region.

pairs of lines passing through two points of S, then $r^{C_i}(S) = r(S)$, where $r(S)$ is the number of non-crossing rays configurations from S studied in previous section, as we have seen there that it suffices to consider canonic sets of rays. Therefore when looking for point sets S and convex regions R_C such that $r^C(S)$ reaches a minimum value, we see that this happens for the same configurations such that $r(S)$ is minimized.

As for an upper bound, since $r^C(S)$ increases the more C tightens around S, in order to obtain a maximum value for $r^C(S)$ we can assume that C is precisely the convex hull of S. A specially interesting case arises when all the points of S are in convex position, i.e., C is a convex polygon with vertices $p_1, \ldots, p_n$ given in clockwise order, and each point p_i is matched with a point q_i on C, and we are interested in counting the possible orders for the points $q_1, \ldots, q_n$. Notice that in this case the number of feasible permutations doesn't change if we replace the polygon C by any convex curve C containing the n points, or if we move the points of S to other positions on C without changing their order. Hence, in this case, $r^C(S)$ only depends on the number of points of the convex set S, and we will use the notation $r^{conv}(n)$ for this amount. The following proposition proves that we can characterize in this case the feasible permutations and count them.

Proposition 4.1. *Let C be the boundary of a convex polygon with vertices $p_1, \ldots, p_n$. A permutation π is feasible as C-matching permutation, if and only if, any five indices $i_1 < i_2 < i_3 < i_4 < i_5$ neither appear in the cyclic order $i_1 i_3 i_5 i_2 i_4$ nor in the cyclic order $i_1 i_4 i_2 i_5 i_3$, and any six indices $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$ do not appear in the cyclic order $i_1 i_4 i_5 i_2 i_3 i_6$. The value of $r^{conv}(n)$ is asymptotically*

$$r^{conv}(n) \approx \frac{125\sqrt{5}}{54\sqrt{\pi}} n^{-3/2} 5^n.$$

When the points of S are in general position, and C is the convex hull of S, we can provide the following bound:

Proposition 4.2. *$r^C(S) \leq 4^n C_n$ for any set S of n points and any convex C enclosing S.*

5 Configurations of non-crossing rays from the vertices of regular polygons

In this final section we study how many different configurations of pairwise non-crossing rays can be obtained when the apices are a point set S consisting of the vertices of a regular n-gon. The number $r(S)$ depends only on n, and will be denoted by $r_{reg}(n)$. We do not know exact formulas for $r_{reg}(n)$, but we have been able to obtain an asymptotic estimate:

Theorem 5.1. *Let $r_{reg}(n)$ be the number of different cyclic permutations that can be obtained shooting sets of pairwise noncrossing rays from the vertices of a regular n-gon. Then*

$$\limsup |r_{reg}(n)|^{1/n} \geq 2.2453.$$

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