Diagonal flips in labelled planar triangulations

Zhicheng Gao *
Jorge Urrutia†‡
Jianyu Wang*

Abstract

A classical result of Wagner states that any two (unlabelled) planar triangulations with the same number of vertices can be transformed into each other by a finite sequence of diagonal flips. Recently Komuro gives a linear bound on the maximum number of diagonal flips needed for such a transformation. In this paper we show that any two *labelled* triangulations can be transformed into each other using at most $O(n \log n)$ diagonal flips. We also show that for planar triangulations a linear number of flips suffice. We will also show that any planar triangulation with $n > 4$ vertices has at least $n - 2$ flippable edges. Finally, we prove that if the minimum degree of a triangulation is at least 4 then it contains at least $2n + 3$ flippable edges. These bounds are tight.

1 Introduction

A planar simple graph T with $n \geq 4$ vertices is called a *planar triangulation* if it has exactly $3n-6$ edges. This terminology follows from the fact that any embedding of any such graph on the plane partitions it into a set of triangular faces, i.e. faces bounded by three edges of T. That the faces of T are well defined follows from a well known result of Whitney [18] that up to isomorphisms maximal planar graphs have a unique embedding on the plane.

This allows us to define the concept of flipping edges on planar triangulations as follows: let v_iv_j be an edge of a planar triangulation T, and $\{v_i, v_j, v_k\}$ and $\{v_i, v_j, v_l\}$ be the vertices of the faces of G containing v_iv_j on their boundaries. We say that v_iv_j is *flippable* if v_k and v_l are not adjacent in T. By flipping v_iv_j, we mean the operation of removing it from T followed by the insertion of v_kv_l into T. It is easy to see that this produces a new graph T' which is also a planar triangulation; see Figure 1. This operation is called a *diagonal flip* on v_iv_j.

A classical result of Wagner states that any two planar *unlabelled* triangulations with the same number of vertices can be transformed into each other by a sequence of diagonal flips.

Let $\mathcal{T}_n$ be the set of all planar *unlabelled* triangulations with n vertices. The *diagonal flip adjacency graph* denoted by $G_{\mathcal{T}}(\mathcal{T}_n)$ is the graph with vertex set $\mathcal{T}_n$, two members of $\mathcal{T}_n$ being adjacent if and only if one can be transformed into other by a single diagonal flip. In this language Wagner's result implies that $G_{\mathcal{T}}(\mathcal{T}_n)$ is connected.

*Department of Mathematics and Statistics, Carleton University, Ottawa, Canada K1S 5B6
†School of Information Technology and Engineering, University of Ottawa, Ottawa, Canada K1N 6N5
‡Instituto de Matemáticas, Universidad Nacional Autónoma de México, México D.F. México

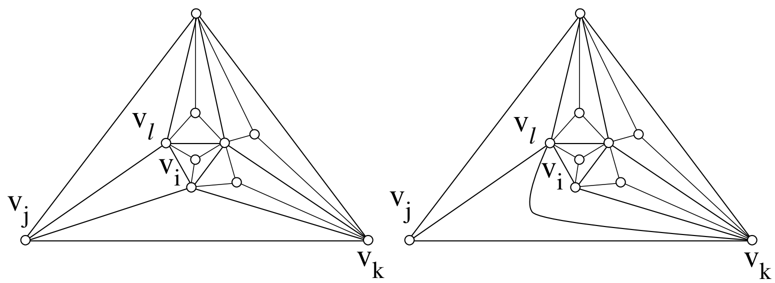


Figure 1: Flipping an edge in a planar triangulation

Let $V = \{v_1, \ldots, v_n\}$ be a set of n labelled vertices and $\mathcal{T}_n$ the set of all planar triangulations on $V = \{v_1, \ldots, v_n\}$. We define $G^l_{\mathcal{T}}(\mathcal{T}_n)$, the graph of labelled triangulations of $V = \{v_1, \ldots, v_n\}$, to be the graph with vertex set $\mathcal{T}_n$ in which two triangulations are adjacent if one can be obtained from the other by means of a diagonal flip.

Dewdney [5], Negami and Watanabe [14] have shown similar results for triangulations of the torus, the projective plane and the Klein bottle. It is easy to see that Wagner's result extends to labelled planar triangulations. However it is not always true for labelled triangulations on the projective plane, and the torus, since there are different triangular embeddings of a labelled complete graph in each of these surfaces. For triangulations in general surfaces, $G_{\mathcal{T}}(\mathcal{T}_n)$ need not be connected even for unlabelled triangulations [12]. However, Negami [13] showed that for any surface Σ, there is a constant L, such that $G^l_{\mathcal{T}}(\mathcal{T}_n)$ is connected for labelled triangulations with at least L vertices. Recently Komuro, Nakamoto and Negami [11] obtained similar results for triangulations with minimum vertex degree at least 4. Diagonal flips preserving some specified properties are discussed in [4].

A closely related subject, i.e. the study of triangulations of point sets has been studied independently by a different group of researchers; see [2, 3, 4, 6, 7, 8, 9, 15]. A triangulation T of a point set P_n is a partitioning of the convex hull of P_n into a set of triangles such that the vertices of these triangles are elements of P_n, and no triangle of T contains an element of P_n in its interior. The edges of the triangles of T are straight line segments joining pairs of elements of P_n; see Figure 2. In this context an edge pq of a triangulation is flippable if the points p and q are contained in the boundary of two triangles such that their union is a convex quadrilateral C; see [8]. By "flipping pq" we understand the operation of removing pq from a triangulation and replacing it by the other diagonal of C to obtain a new triangulation of P_n; see Figure 2.

The graph of triangulations $GT(P_n)$ of a point set P_n is the graph whose vertex set is the set of triangulations of P_n. Two triangulations are adjacent if one can be obtained from the other by an edge flip. It is known that the graph of triangulations of a point set is connected, and that there are collections of points P_n such that the diameter of $GT(P_n)$ is $O(n^2)$; see [8]. In the same paper, it is shown that any triangulation of a point set contains at least $\lceil \frac{n-4}{2} \rceil$ flippable edges.

It is also well known that rooted triangulations of polygons correspond to rooted binary trees and that diagonal flips in such triangulations correspond to rotations in the corresponding binary trees. Let $\mathcal{C}_n$ be the set of all triangulations of a convex n-gon. Sleator, Tarjan and Thurston [16]

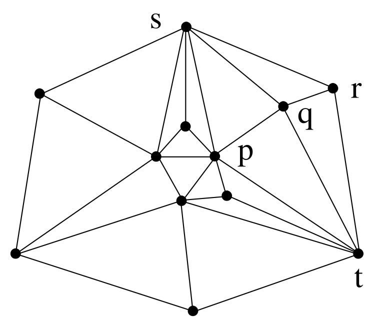


Figure 2: A triangulation T of a point set. Edge pq is flippable in T; edge qr is not.

proved that the diameter of the diagonal flip adjacency graph of $\mathcal{C}_n$ is $2n - 10$ for all sufficiently large n.

At this point, we point out that Wagner's original argument for unlabelled planar triangulations gives a quadratic bound on the diameter of $G_{\mathcal{T}}(\mathcal{T}_n)$. That the diameter of $G_{\mathcal{T}}(\mathcal{T}_n)$ is linear was recently proved by Komuro [10]. The difference in the diameter of $G_{\mathcal{T}}(\mathcal{T}_n)$ and the diameter of graph of triangulations of point sets leads in a natural way to the study of $G^l_{\mathcal{T}}(\mathcal{T}_n)$ for labelled graphs: the positions of the elements of a point set make them *labelled* points in a natural way. To be more precise, in this paper we study the following problem. Let $V = \{v_1, \ldots, v_n\}$ be a set of vertices, and G and G' be two planar triangulations with vertex set V. How many edge flips are needed to transform G into G'? In Figure 3 we show two labelled triangulations on $\{v_1, \ldots, v_6\}$ such that to transform one into the other requires 2 flips. Notice that as unlabelled triangulations, the triangulations shown in the same figure are isomorphic, and no flipping is needed to transform one into the other, however as labelled triangulations they are different.

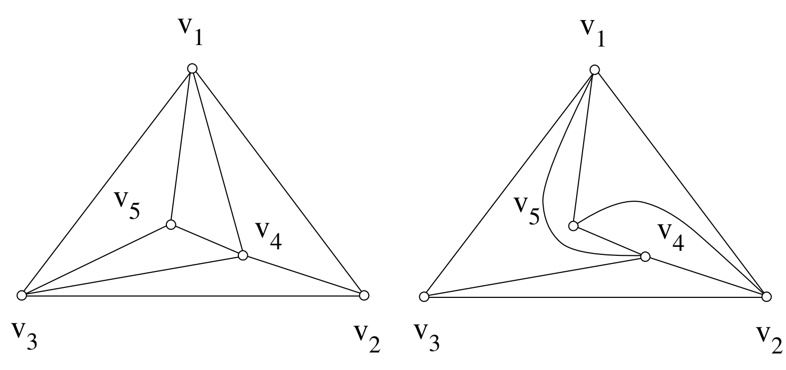


Figure 3: Two labelled triangulations at distance 2

In this paper we prove that any labelled triangulation on n vertices can be transformed into any other labelled triangulation using at most $O(n \log n)$ flips. To end this paper, we prove that that any planar triangulation with n vertices contains at least $n-2$ flippable edges, and that this bound is tight. We point out that using results proved by Ando and Komoro [1] it is easy to prove that any planar triangulation with n vertices contains at least $\lceil \frac{n}{2} \rceil$ flippable edges. In the rest of this paper, all triangulations considered will be assumed to be labelled triangulations on $V = \{v_1, \ldots, v_n\}$.

2 The diameter of $G^l(\mathcal{T}_n)$

Given two vertices v_i and v_j, a triangulation T will be called a $\Delta(i,j)$ triangulation if v_i and v_j are both adjacent to all vertices of T; see Figure 4. We now prove the following result that mirrors Lemma 2 in Komuro's paper [10]:

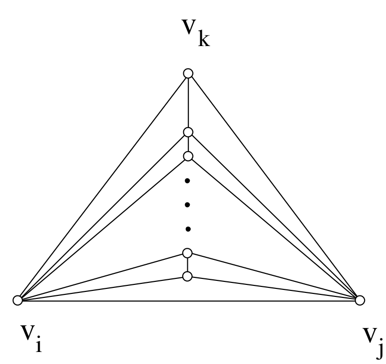


Figure 4: A $\Delta(i,j)$ triangulation.

Theorem 1 *Let T be a triangulation and let v_i, v_j be adjacent vertices in T. Then we can transform T into a $\Delta(i,j)$ triangulation with at most $4n-16$ diagonal flips. Moreover let F be one of the triangular faces of T containing edge v_iv_j. Then $\Delta(i,j)$ can be chosen such that F is also a face in $\Delta(i,j)$ and the edges bounding F are never flipped.*

Proof: To prove our result, we use the following concept introdced by Komuro [10]. Let F be one of the two faces of T containing edge v_i v_j, and let v_k be the third vertex of F. We define the *potential* of T by

$$p_T(i,j) = 3\deg(v_i) + \deg(v_j),$$

and show that if T is not a $\Delta(i,j)$ triangulation, then by performing some diagonal flips we can increase $p_T(i,j)$. Note that $p_T(i,j) \leq 4(n-1)$ with the equality holding only when $\deg(v_i) = \deg(v_j) = n-1$, i.e. T is a $\Delta(i,j)$ triangulation.

Let $v_j, v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(l-1)}, v_{\sigma(l)} = v_k$ be the neighbors of v_i in T in the anticlockwise order. For convenience, we set $v_{\sigma(0)} = v_i$. Let m be the largest integer such that $v_{\sigma(1)}, v_{\sigma(2)}, \ldots v_{\sigma(m)}$ are all adjacent to v_j in T, and each triangle $v_jv_{\sigma(i-1)}v_{\sigma(i)}$ bounds a face for $i = 1, 2, \ldots, m$. If $m = l$, then T is a $\Delta(i,j)$ triangulation and no diagonal flip is needed. Otherwise let $v_jv_{\sigma(m)}u$ be the other face incident with the edge $v_jv_{\sigma(m)}$ with $u \neq v_{\sigma(m+1)}$. We distinguish the following two cases.

CASE **1:** u is not a neighbor of v_i. If $m = 1$ we can flip $v_j v_{\sigma(1)}$, and increase $p_T(i,j)$ by 2. If $m > 1$, we can flip $v_j v_{\sigma(m)}$ and then $v_{\sigma(m-1)} v_{\sigma(m)}$, and increase $p_T(i,j)$ by 2.

CASE **2:** $u = v_{\sigma(s)}$ for some $m + 2 \leq s \leq l$. In this case $v_{\sigma(m)} v_{\sigma(s)}$ can be flipped and $p_T(i,j)$ increases by 1.

By iterating the above process, we transform T into a $\Delta(i,j)$ triangulation in which F remains a face. Notice that the total number of diagonal flips involved does not exceed $4(n-1) - p_T(i,j)$, and that the edges bounding F are never flipped. Our result follows. ∎

Let T be a $\Delta(i,j)$ triangulation. Notice that $T - \{v_i, v_j\}$ is a path P. Assume without loss of generality that the vertices of P are labelled $\{v_{\sigma(1)}, \ldots, v_{\sigma(n-2)}\}$. If the elements of P are such that $\sigma(1) < \sigma(2) < \ldots < \sigma(n-2)$ we say that T is *sorted*. A $\Delta(i,j)$ triangulation T' is called a *transpose* of T if $T' - \{v_i, v_j\}$ is a path P' obtained from P by transposing two consecutive vertices of P; see Figure 5.

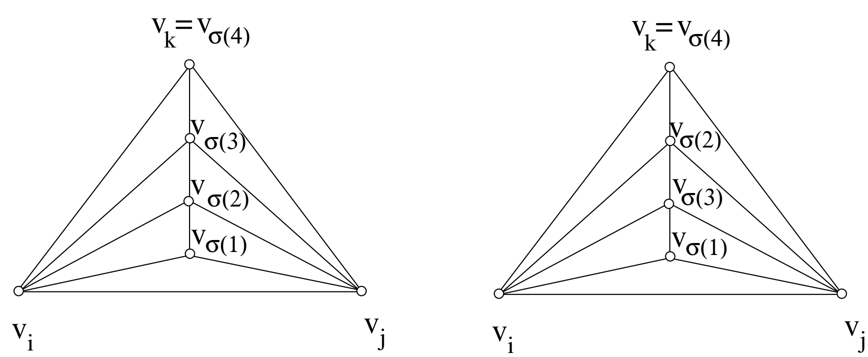


Figure 5: A $\Delta(i,j)$ triangulation and a transposition of it.

The next lemma is easy to prove:

Lemma 1 *Let T be a $\Delta(i,j)$ triangulation, and T' a transposition of T. Then T' can be obtained from T by flipping at most 4 edges.*

A $\Delta(1,2)$ triangulation such that the vertices of one of its faces are precisely v_1, v_2, v_n will be called a *normal* triangulation. We now prove:

Lemma 2 *Within $O(n)$ diagonal flips, any planar triangulation T with n vertices can be transformed into a normal $\Delta(1,2)$ planar triangulation T'.*

Proof: Let v_i be any neighbour of v_1. By Theorem 1 we can transform this triangulation into a $\Delta(1,i)$ triangulation using $O(n)$ diagonal flips. Since vertex v_2 is adjacent to v_1, again by Theorem 1, we can transform this triangulation into a $\Delta(1,2)$ triangulation with a linear number of flips. Using Lemma 1, we can now perform a linear number of transpositions until v_1, v_2 and v_n belong to the same face. ∎

Notice that using this lemma, together with Lemma 1, we can prove that any planar triangulation can be transformed to the sorted $\Delta(1,2)$ triangulation with a quadratic number of transpositions, i.e. a quadratic number of flips. We now proceed to show how to accomplish this in at most $O(n \log n)$ flips.

2.1 The binary triangulation

To achieve our goal, we define a special type of triangulations which we call *binary triangulations.* A planar triangulation with vertex set $\{v_1, \ldots, v_n\}$ is called binary if:

1. The vertices of a face of T are v_1, v_2, v_n
2. The dual graph of $T - \{v_n\}$ (excluding the vertex corresponding to the only face of $T - \{v_n\}$ that is not a triangular face) is an almost balanced binary tree, i.e. it is a tree obtained by removing some leaves from a balanced binary tree; see Figure 6.

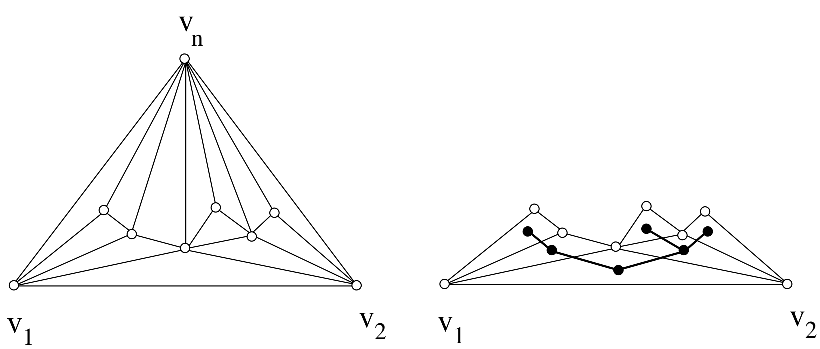


Figure 6: A binary triangulation.

We now proceed to show how a binary triangulation can be transformed into the sorted normal triangulation in at most $O(n \log n)$ flips.

A 2-Δ *triangulation* is a triangulation consisting of two sorted triangulations $\Delta(i,j)$ and $\Delta(j,k)$, having two common vertices, v_j and v_s glued along the edge joining v_j to v_s plus the edge joining v_i to v_k. We further require s to be the largest subindex of all the vertices of the 2-Δ triangulation; see Figure 7(a).

The following lemma will be essential to prove our main result:

Lemma 3 *Any* 2-Δ *triangulation can be transformed into a sorted triangulation with a linear number of flips.*

Proof: Let $\Delta(i,j)$ and $\Delta(j,k)$ be the sorted triangulations forming T, and let v_s be the common neighbour of v_i and v_k, as in Figure 7(a). Let $v_{\alpha(1)}, \ldots, v_{\alpha(r)}$ and $v_{\beta(1)}, \ldots, v_{\beta(t)}$ be the vertices in $\Delta(i,j)$ and $\Delta(j,k)$ respectively, different from v_i, v_j, v_k, and v_s.

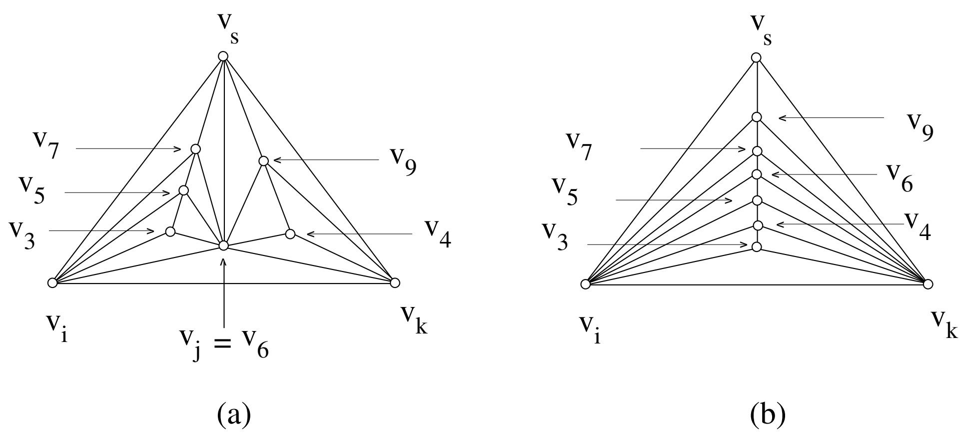


Figure 7: A 2 - Δ triangulation and the resulting merged triangulation.

Assume without loss of generality that $\alpha(1) < \beta(1)$. Then by performing two flips, we can obtain a triangulation in which $v_{\alpha(1)}$ is adjacent to v_i, v_j and v_k; see Figure 8(b). It is now easy to see that using three flips at a time, we can move the remaining vertices of $\Delta(i,j)$ and $\Delta(j,k)$ so that in the end we get an almost sorted $\Delta(i,k)$ triangulation; see Figure 8(c),(d). The only vertex out of place is perhaps v_j. This can be fixed by performing a linear number of transpositions until v_j moves to its correct position. ∎

We are ready to prove:

Lemma 4 *Let T be a binary planar triangulation with n vertices such that the vertices of the external face are v_1, v_2, and v_n. Then T can be transformed into the $\Delta(1,2)$ sorted triangulation by performing $O(n \log n)$ diagonal flips.*

Proof: Our theorem is true for $2^2 \leq n \leq 2^3$. Suppose that our result is true for $2^{i-1} \leq n \leq 2^i$. We now show that it also holds for $2^i \leq n \leq 2^{i+1}$. Let $2^i \leq n \leq 2^{i+1}$ and let T be a binary triangulation with n vertices. Observe that T splits into two binary triangulations T' and T'' with n_1 and n_2 vertices, $2^{i-1} \leq n_1, n_2 \leq 2^i$. By induction on i, T' and T'' can be transformed into sorted Δ triangulations in $O(n_1 \log n_1)$ and $O(n_2 \log n_2)$ flips. By Lemma 3 we can transform the resulting triangulation into a sorted $\Delta(1,2)$ triangulation with a linear number of flips. Our result now follows . ∎

We proceed to prove our main result:

Theorem 2 *Let T and T' be any two labelled planar triangulations. Then T can be transformed into T' using $O(n \log n)$ flips, i.e. the diameter of $G^l(\mathcal{T}_n)$ is at most $O(n \log n)$.*

Proof: To prove our result it is enough to show that T can be transformed to the sorted $\Delta(1,2)$ triangulation using $O(n \log n)$ flips. By Lemma 2, within $O(n)$ diagonal flips we can transform T

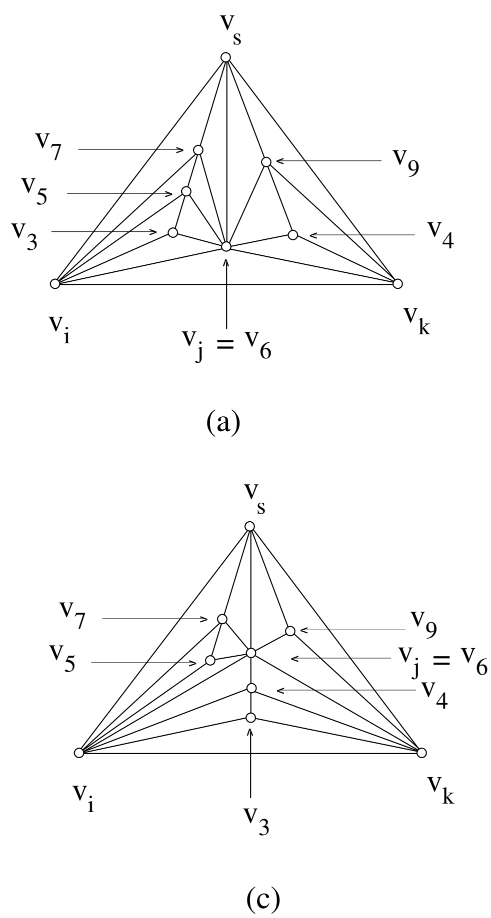


Figure 8: Illustrating Lemma 3.

into a not necessarily sorted $\Delta(1,2)$ triangulation T_1 whose three exterior vertices are v_1, v_2, v_n. 'Next by Theorem 1, we can transform T_1 into a binary triangulation T_2 using $O(n)$ flips. Finally by Lemma 3, we can transform T_2 into the sorted $\Delta(1,2)$ triangulation. Our result follows. ∎

3 The minimum vertex degree of $G^l_{\mathcal{T}}(\mathcal{T}_n)$

Any planar triangulation contains a large number of flippable edges. In [8] it is proved that any triangulation of a set of n points in general position contains at least $\lceil \frac{n-4}{2} \rceil$ flippable edges. In this section we give a tight bound on the number of flippable edges of planar triangulations, namely we prove:

Theorem 3 *Any planar triangulation T with $n > 4$ vertices contains at least $n-2$ flippable edges. If T has minimum vertex degree at least 4, then T contains at least $\min\{2n+3, 3n-6\}$ flippable edges. Our bounds are tight.*

Proof: A triangle in a triangulation T is called *separating* if there are vertices inside as well as outside the triangle. Two edges are called *cofacial* if they belong to the boundary of a face of T. Let F ($\bar{F}$) be the set of flippable (nonflippable) edges in T. Define a relation $\mathcal{R} \subseteq \bar{F} \times F$ as follows:

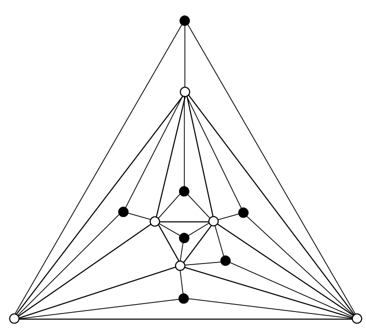
$$(e, f) \in \mathcal{R} \iff e \in \bar{F}, f \in F, \text{ and } e \text{ and } f \text{ are cofacial.}$$

We claim that each nonflippable edge is related to *at least two* flippable edges. Let $e = v_iv_j$ be any nonflippable edge in T, and let $\{v_i, v_j, v_k\}$ and $\{v_i, v_j, v_l\}$ be the vertices of the two triangular faces of T incident with v_iv_j. Since v_iv_j is nonflippable, v_k and v_l are adjacent in T. Since T has more than four vertices, vertices v_i and v_j cannot both have degree 3. If vertex v_i has degree at least 4, then both edges v_iv_k and v_iv_l are flippable; if vertex v_j has degree at least 4, then both edges v_jv_k and v_jv_l are flippable. On the other hand, each flippable edge is incident with exactly two faces, and hence is related to *at most four* nonflippable edges. Therefore we have:

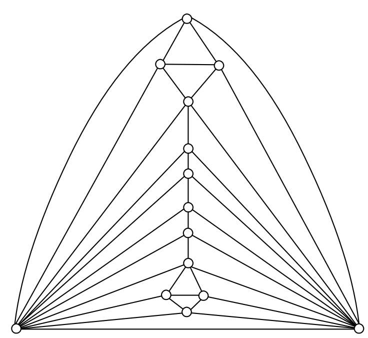
$$2|\bar{F}| \leq |\mathcal{R}| \leq 4|F|.$$

Since the total number of edges in T is $3n - 6$, it follows that the number of flippable edges is at least $(3n - 6)/3 = n - 2$.

Examples of planar triangulations that acheive this bound can be constructed as follows: Let T' be any planar triangulation with m vertices. Thus T' contains $2m - 4$ triangular faces. Let T be the triangulation obtained as follows: In the middle of each of these triangular faces, insert a vertex adjacent to the vertices of the face. See Figure 9(a). It is easy to see that the only edges of T that are flippable, are exactly the edges of T', i.e. $3m - 6$ edges. On the other hand T contains exactly $m + 2m - 4 = 3m - 4$ vertices. Taking $n = 3m - 4$ yields the desired result. This proves the first part of our theorem.



(a)



(b)

Figure 9: Two triangulations, the first with $n - 2$ flippable edges, and the second with minimum degree 4 and $2n + 3$ flippable edges.

The argument in the previous paragraph shows that if T has more than four vertices and a nonflippable edge, then T contains a separating triangle. Thus the second part of our result holds if T contains no separating triangles. Assume that T has minimum vertex degree at least 4 and T contains a separating triangle. The above argument shows that each nonflippable edge is related to exactly four flippable edges, i.e. $4|\bar{F}| = |\mathcal{R}|$. Also, if $\{v_i, v_j, v_k\}$ are the vertices of a face of

T such that $v_i v_j$ is a flippable edge, then at least one of $v_i v_k$ and $v_j v_k$ is flippable. (Otherwise, the above argument implies that v_k has degree 3.) Hence each flippable edge is related to *at most two* nonflippable edges. Now we show that T contains at least 18 flippable edges which are not related to any nonflippable edge, and there are at least 3 extra flippable edges which are related to at most one nonflippable edge. Let $v_i v_j v_k$ be a separating triangle in T such that the triangulation $T(v_i v_j v_k)$, which consists of the triangle $v_i v_j v_k$ and all its interior vertices, contains no separating triangles. Since T has no vertex of degree 3, $T(v_i v_j v_k)$ contains at least 6 vertices. Since $T(v_i v_j v_k)$ contains no separating triangle, all edges of $T(v_i v_j v_k)$ are flippable in T. Therefore, all edges inside $v_i v_j v_k$ (there are least 9 such edges) are not related to any nonflippable edges. Similarly, T contains at least 9 flippable edges outside $v_i v_j v_k$ which are not related to any nonflippable edge. Notice also that each of the three edges $v_i v_j,\ v_j v_k,\ v_i v_k$ is related to at most one nonflippable edge. Thus we obtain:

$$4|\bar{F}| = |\mathcal{R}| \leq 2(|F| - 18 - 3) + 3.$$

Using $|\bar{F}| + |F| = 3n - 6$, we obtain $|F| \geq 2n + 2 + 1/2$, i.e. $|F| \geq 2n + 3$.

Triangulations that achieve this bound can be obtained as follows. Let T' be a $\Delta(i, j)$ triangulation with $n - 6$ vertices. Insert a triangle in each of the two faces incident with v_i and v_j in such a way that the degree of the six new vertices is four; see Figure 9(b). The reader can easily verify that the resulting triangulation achieves the previous bound. Our result follows. ∎

4 Conclusions and open problems

In this paper we proved that any planar labelled triangulation can be transformed into any other labelled planar triangulation performing $O(n \log n)$ flips. We do not know if this bound is optimal. Neverteless we conjecture:

Conjecture 1 *There are labelled triangulations T and T' on n vertices such that to transform T into T' requires $O(n \log n)$ flips.*

We also proved that any planar triangulation with n vertices contains at least $n - 2$ flippable edges, and that any triangulation with minimum degree 4 contains at least $2n + 3$ flippable edges. These bounds are tight.

References

[1] Ando, K., and H. Komuro,"The seven graphs whose H-transformations are uniquely determined", (1997) preprint.

[2] B. Aronov, R. Seidel and D. Souvaine, "On Compatible triangulations of simple polygons", *Computational Geometry: Theory and Applications*— **3** (1993) 27-35.

[3] Bern, M. and D. Eppstein, "Mesh generation and optimal triangulation", in *Computing in Euclidean Geometry*, D.Z. Du and F. K. Hwang, eds., World Scientific, 1992, 23-90.

[4] R. Brunet, A. Nakamoto and S. Negami, “Diagonal flips of triangulations on closed surfaces preserving specified properties”, *J. Combin. Theory Ser. B* **68**, (1996), no. 2, 295–309.

[5] A. K. Dewdney, “Wagner’s theorem for the torus graph”, *Discrete Math.* **4** (1973), 139-149.

[6] Fortune, S., “Voronoi diagrams and Delaunay triangulations”, in *Computing in Euclidean Geometry*, D. Z. Du and F. K. Hwang eds., World Scientific, 1992, 193-234.

[7] L. Guibas, J. Hershberger, D. Levem, M. Sharir and R. Tarjan, “Algorithms for visibility and shortest path problems inside triangulated simple polygons”, *Algorithmica*, **2** (1987), 200 - 237.

[8] F. Hurtado, M. Noy and J. Urrutia, “Flipping edges on triangulation”, Proc. 12nd ACM Symposium on Computational Geometry, May 24-26 1996, 214-223.

[9] E. Kranakis, and J. Urrutia, “Isomorphic triangulations with a small number of Steiner points”. To appear in *Int. Journal on Computational Geometry.*

[10] Hideo Komuro, “The diagonal flips of triangulations on the sphere”, *Yokohama Math. J.* **44** (1997), 115-122.

[11] H. Komuro, A. Nakamoto and S. Negami, “Diagonal flips in triangulations on closed surfaces with minimum degree at least 4”, to appear in *J. Combin. Theory Ser. B.*

[12] S. Lawrencenko, S., Negami and A.T. White, “Three nonisomorphic triangulations of an orientable surface with the same complete graph”, *Discrete Math.* **135** (1994), 367–369.

[13] S. Negami, “Diagonal flips in triangulations of surfaces, *Discrete Math.* **135** (1994), 225–232.”

[14] S. Negami and S. Watanabe, “Diagonal transformations of triangulations on surfaces”, *Tsukuba J. Math.* **14** (1990), 155-166.

[15] Saalfeld, A., “Joint triangulations and triangulated maps”:, in Proc. 3rd Annual ACM Symposium in Computational Geometry, (1987) 195-204.

[16] D. Sleator, R. Tarjan and W. Thurston, “Rotation distance, triangulations, and hyperbolic geometry”, *J. Amer. Math. Soc.* **1** (1988), 647-681.

[17] K. Wagner, “Bemerkungem zum Vierfarbenproblem”, *Jber. Deutsch. Math-Verein* **46** (1936), 126-132.

[18] H. Whitney, “Congruent graphs and the connectivity of graphs”, *Amer. J. Math.* **54** (1932), 150-168.