Geometric achromatic and pseudoachromatic indices

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Abstract

The pseudoachromatic index of a graph is the maximum number of colors that can be assigned to its edges, such that each pair of different colors is incident to a common vertex. If for each vertex its incident edges have different color, then this maximum is known as achromatic index. Both indices have been widely studied. A geometric graph is a graph drawn in the plane such that its vertices are points in general position, and its edges are straight-line segments. In this paper we extend the notion of pseudoachromatic and achromatic indices for geometric graphs, and present results for complete geometric graphs. In particular, we show that for n points in convex position the achromatic index and the pseudoachromatic index of the complete geometric graph are $\lfloor \frac{n^2+n}{4} \rfloor$.

1 Introduction

A *vertex coloring* of a simple graph G with k colors, is a surjective function that assigns to each vertex of G a color from the set $\{1, 2, \ldots, k\}$. A coloring of a graph is *proper* if any two adjacent vertices have different color, and it is *complete* if every pair of colors appears on at least one pair of adjacent vertices. The *chromatic number* $\chi(G)$ *of* G is the smallest number k for which there exists a proper coloring of G using k colors. It is not hard to see that any proper coloring of G with $\chi(G)$ colors is a complete coloring. The *achromatic number* $\alpha(G)$ of G is the largest number k for which there exists a proper and complete coloring of G using k colors [18]. The *pseudoachromatic number* $\psi(G)$

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of G is the largest number k for which there exists a complete coloring of G using k colors [17].

The *chromatic index* $\chi_1(G)$, *achromatic index* $\alpha_1(G)$ and *pseudoachromatic index* $\psi_1(G)$ of G, are defined respectively as the chromatic number, achromatic number and pseudoachromatic number of the line graph $L(G)$ of G. Notationally, $\chi_1(G) = \chi(L(G))$, $\alpha_1(G) = \alpha(L(G))$ and $\psi_1(G) = \psi(L(G))$.

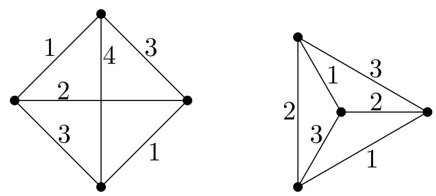
A central topic in Graph Theory is to study the behavior of any parameter in complete graphs. For instance, the authors of [6] determined the exact values of α_1 for some complete graphs. Some of the authos of this paper determined the exact value of α_1 and ψ_1 for some complete graphs, see [3, 4]. As a natural extension of this work, in this paper we introduce the notion of pseudoachromatic and achromatic indices to geometric graphs, and present upper and lower bounds for the case of complete geometric graphs.

A *geometric graph* is a graph drawn in the plane such that its vertices are points in general position, and its edges are straight-line segments. A geometric graph is complete if its edge set contains every segment connecting a pair of vertices. The problem of determining the chromatic, achromatic, and pseudoachromatic indices for complete geometric graphs has a different nature than the same problem in graphs. To gain some intuition about this difference, consider the two geometric graphs shown in Figure 1. Each of these graphs is complete and has four vertices. For geometric graphs, one can define the incidences between edges in various ways. For instance, we can say that two edges are adjacent if they have a common endpoint or they cross each other. Under this setting, the chromatic number of the graph shown in Figure 1(a) is 4, while the chromatic number of the graph shown in Figure 1(b) is 3. In contrast, as is well known, $\chi_1(K_4) = 3$ (by K_n we denote the complete graph on n vertices).

In the next section we formally define chromatic, achromatic and pseudoachromatic indices for geometric graphs, provide some basic relations between them, and revise previous related results. In Section 3 we consider complete geometric graphs defined by point sets in convex position in the plane, and determine the exact value of the achromatic and pseudoachromatic indices. Finally, in Section 4, we consider complete geometric graphs defined by point sets in general position in the plane, and present lower and upper bounds for the geometric pseudoachromatic index.

2 Preliminaries

Throughout this paper we assume that all sets of points in the plane are in general position, that is, no three points are on a common line. Let $G = (V, E)$ be a simple graph. A *geometric embedding* of G is an injective map that sends V to a set S of points in the plane, and E to a set of (possibly crossing) straight-line segments whose endpoints belong to S. A *geometric graph* G is the image of a particular geometric embedding of G. For brevity we refer to the points in S as vertices of G, and to the straight-line segments connecting two points in S as edges of G. Note that any set of points in the plane induces a complete



(a) $\chi_1(\mathsf{K_4}) = 4$. (b) $\chi_1(\mathsf{K_4}) = 3$

Figure 1: There are only two combinatorially different geometric graphs of K_4. Two edges are adjacent if they intersect. The geometric chromatic index χ_g of K_4 is 4.

geometric graph.

For geometric graphs, it is possible to define edge adjacencies in different ways. We say that two edges *intersect* if they have a common endpoint or they cross each other (i.e. they have an interior point in common). Two edges are *disjoint* if they do not intersect. Depending on the type of adjacency that we consider, four different families of geometric graphs can be defined: those in which the edges intersect, and their complement; or those in which the edges cross, and their complement.

Let G be a geometric graph defined from any of the four adjacency criteria described above. A coloring of the edges of G is *proper* if every pair of adjacent edges have different color. A coloring is *complete* if each pair of colors appears on at least one pair of adjacent edges.

The *chromatic index* $\chi_1(\mathsf{G})$ of G is the smallest number k for which there exists a proper coloring of the edges of G using k colors. Figure 1 shows the chromatic index of two geometric graphs of K_4, where two edges are adjacent if they intersect. The *achromatic index* $\alpha_1(\mathsf{G})$ of G is the largest number k for which there exists a complete and proper coloring of the edges of G using k colors. The *pseudoachromatic index* $\psi_1(\mathsf{G})$ of G is the largest number k for which there exists a complete coloring of the edges of G using k colors.

We extend these definitions to graphs in the following way. Let G be a graph. The *geometric chromatic index* $\chi_g(G)$ of G is the largest value k for which a geometric graph H of G exists, such that $\chi_1(\mathsf{H}) = k$; see Figure 1. Likewise, the *geometric achromatic index* $\alpha_g(G)$ and the *geometric pseudoachromatic index* $\psi_g(G)$ of G, are defined as the smallest value k for which a geometric graph H of G exists such that $\alpha_1(\mathsf{H}) = k$ and $\psi_1(\mathsf{H}) = k$, respectively.

The chromatic index of some geometric graphs has been studied before. Let S be a point set in convex position, and let $\mathsf{K_n}$ be the complete geometric graph defined by S. Consider the graph D_n whose vertices are the edges of $\mathsf{K_n}$, two of which are adjacent if the corresponding edges are disjoint. The problem of determining the chromatic number of these types of graphs was defined in [2], where the authors gave some upper and lower bounds for $\chi(D_n)$. Later the bounds were improved in [10] and [13]; finally in [20] it was proved that

$\chi(D_n) = n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor$. Now consider the case when the point set S is in general position, and let l be a positive integer. The graph $I(S)$, is the graph whose vertices are the subsets of S with l elements, two of which are adjacent if the convex hulls of the respective subsets intersect. This graph was defined in [2], where the authors study its chromatic number for the case when $l = 2$. If we denote by K_n the complete geometric graph with vertex set S and say that two edges are adjacent if they intersect, then for the case $l = 2$, $\chi(I(S)) = \chi_1(\mathsf{K}_n)$. In [2] the authors prove that $\chi_1(\mathsf{K}_n) = n$ and that $n \leq \chi_g(K_n) \leq cn^{3/2}$ for some constant $c > 0$ (by K_n we denote the complete graph on n vertices). The other adjacency criteria have also been considered. Basic bounds for the chromatic number in these cases can be found in [15]. To the best of our knowledge, these are the only results available in the literature concerning coloring of geometric graphs.

In this paper, we consider the adjacency criterion in which two edges are adjacent if they intersect. We prove the following.

Theorem 2.1. *i) For each $n \neq 4$, and the vertices of K_n in convex position*

$$\alpha_1(\mathsf{K}_n) = \psi_1(\mathsf{K}_n) = \lfloor \tfrac{n^2+n}{4} \rfloor,$$

ii) For each $n > 18$,

$$0.0710n^2 - \Theta(n) \leq \psi_g(K_n) \leq 0.1781n^2 + \Theta(n).$$

Before we proceed, we present some basic relations between the different chromatic indices that we have defined above. For graphs we obtain:

$$\chi_1(G) \leq \chi_g(G) \tag{2.1}$$

$$\chi_1(G) \leq \alpha_1(G) \leq \psi_1(G) \leq \psi_g(G) \tag{2.2}$$

$$\alpha_g(G) \leq \psi_g(G). \tag{2.3}$$

For geometric graphs we obtain:

$$\chi_1(\mathsf{G}) \leq \alpha_1(\mathsf{G}) \leq \psi_1(\mathsf{G}). \tag{2.4}$$

Consider the cycle C_n of length $n \geq 3$. In this case $\chi_1(C_n)$ is equal to 2 if n is even, and it is equal to 3 if n is odd. On the other hand, it is not hard to see that $\chi_g(C_n) = n - 1$ if n is even and $\chi_g(C_n) = n$ if n is odd. However, $\alpha(C_n) = \alpha_1(C_n) = \alpha_g(C_n) = \max\{k\colon k\lfloor \frac{k}{2} \rfloor \leq n\} - s(n)$, where $s(n)$ is the number of positive integer solutions to $n = x^2 + x + 1$. Also, $\psi(C_n) = \psi_1(C_n) = \psi_g(C_n) = \max\{k\colon k\lfloor \frac{k}{2} \rfloor \leq n\}$. These results can be found in [9, 19, 23].

It is known that if G is a planar graph then there always exists a geometric embedding j, where no two edges of $j(G)$ intersect, except possibly in a common endpoint [14]. Therefore, $\psi_1(G) = \psi_1(j(G)) = \psi_g(G)$ and $\alpha_1(G) = \alpha_1(j(G)) \geq \alpha_g(G)$. However, $\chi_1(G) = \chi_1(j(G)) \leq \chi_g(G)$ (for instance, as we mentioned before, $\chi_1(C_4) = 2$ and $\chi_g(C_4) = 3$).

3 Points in convex position

In this section we prove Claim i) of Theorem 2.1: For each $n \neq 4$, and the vertices of K_n in convex position, $\alpha_1(\mathsf{K}_n) = \psi_1(\mathsf{K}_n) = \lfloor \frac{n^2+n}{4} \rfloor$. In Subsection 3.1 we present an upper bound for $\psi_1(\mathsf{G})$ for any geometric graph G. Then in Subsection 3.2 we exclusively work with geometric graphs induced by point sets in convex position, and by giving a proper and complete coloring of any graph in this family, derive a tight lower bound for $\alpha_1(\mathsf{K}_n)$.

3.1 Upper bound: $\psi_1(\mathsf{G}) \leq \lfloor \frac{n^2+n}{4} \rfloor$

The following theorem was shown in [12].

Theorem 3.1. *Any geometric graph with n vertices and $n+1$ edges, contains two disjoint edges.*

Using this theorem we obtain the following result, where the order of a graph denotes the number of its vertices.

Corollary 3.2. *Let G be a geometric graph of order n. There are at most n chromatic classes of size one in any complete coloring of G.*

This corollary immediately implies an upper bound on $\psi_1(\mathsf{G})$.

Theorem 3.3. *Let G be a geometric graph of order n. The pseudoachromatic index $\psi_1(\mathsf{G})$ of G is at most $\lfloor \frac{n^2+n}{4} \rfloor$.*

Proof. We proceed by contradiction. Assume there exists a geometric graph G for which a complete coloring using $\lfloor \frac{n^2+n}{4} \rfloor + 1$ colors exists. This coloring must have at most $\binom{n}{2} - \left(\lfloor \frac{n^2+n}{4} \rfloor + 1 \right)$ chromatic classes of cardinality larger than one. Thus, there are at least $\lfloor \frac{n^2+n}{4} \rfloor + 1 - \left(\binom{n}{2} - \lfloor \frac{n^2+n}{4} \rfloor - 1 \right)$ chromatic classes of size one, that is:

$$1 - \left(\binom{n+1}{2} - 2 \left\lfloor \frac{\binom{n+1}{2}}{2} \right\rfloor \right) + n + 1 = \begin{cases} n+1 & \text{if } \binom{n+1}{2} \text{ is odd,} \\ n+2 & \text{if } \binom{n+1}{2} \text{ is even.} \end{cases} \tag{3.1}$$

This contradicts Corollary 3.2 and therefore the theorem follows. □

3.2 Tight lower bound: $\alpha_1(\mathsf{G}) \geq \lfloor \frac{n^2+n}{4} \rfloor$

In this subsection we prove that the bound presented in Theorem 3.3 is tight. By Inequality 2.4 we know that $\alpha_1(\mathsf{G}) \leq \psi_1(\mathsf{G})$. Now we prove that there exists an infinite family of geometric graphs G, for which $\alpha_1(\mathsf{G}) = \psi_1(\mathsf{G}) = \lfloor \frac{n^2+n}{4} \rfloor$. To derive the lower bound, we exhibit a proper and complete coloring for complete geometric graphs induced by a set of points in convex position. We call this type of graph a *complete convex geometric graph*. The crossing pattern of the

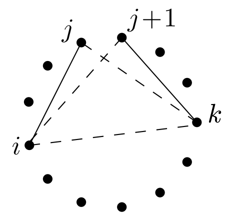


Figure 2: Example for $n = 13$. Edges of the halving pair $(e_{i,j}, e_{j+1,k})$ are shown solid, dashed edges represent the possible halving edges, of which $e_{i,k}$ is a halving edge (the witness) in the shown example. The convex hull of a pair $(e_{i,j}, e_{j+1,k})$ is always a quadrilateral.

edge set of a complete convex geometric graph depends only on the number of vertices, and not on their particular position. Without loss of generality, we therefore assume that the point set of the graph corresponds to the vertices of a regular polygon. In the remainder of this section we exclusively work with this type of graphs.

To simplify the proof of the main statement of this section, in the following we will define different sets of edges and prove some important properties of these sets.

Let G be a complete convex geometric graph of order n, and let $\{1, \ldots, n\}$ be the vertices of the graph listed in clockwise order. For the remainder of this subsection it is important to bear in mind that all sums are taken modulo n; for the sake of simplicity we will avoid writing this explicitly. We denote by $e_{i,j}$ the edge between the vertices i and j. We call an edge $e_{i,j}$ a *halving edge* if in both of the two open semi-planes defined by the line containing $e_{i,j}$, there are at least $\lfloor \frac{n-2}{2} \rfloor$ points of G. Using this concept we obtain the following definition.

Definition 3.4. Let $i, j, k \in \{1, \ldots, n\}$, such that $e_{i,j}$ and $e_{j+1,k}$ do not intersect. We call a pair of edges $(e_{i,j}, e_{j+1,k})$ a *halving pair of edges* (*halving pair*, for short) if at least one of $e_{i,j+1}$, $e_{i,k}$, or $e_{j,k}$ is a halving edge. This halving edge is called the *witness* of the halving pair.

See Figure 2 for an example of a halving pair $(e_{i,j}, e_{j+1,k})$, with $e_{i,k}$ as witness. Note that a halving pair may have more than one witness.

We say that an edge e intersects a pair of edges (f, g) if e intersects at least one of f or g. We say that two pairs of edges intersect if there is an edge in the first pair which intersects the second pair.

Lemma 3.5. *Let G be a complete convex geometric graph of order n. i) Each two halving edges intersect. ii) Any halving edge intersects any halving pair of edges. iii) Any two halving pairs intersect.*

Proof. To prove Claim i) assume that there are two halving edges which do not intersect. These edges divide the set of vertices of G into two disjoint sets of size at least $\lfloor \frac{n-2}{2} \rfloor$ and one set of size at least 4 (the vertices of the two halving edges). Then, the total number of vertices is:

$$2\left\lfloor \frac{n-2}{2} \right\rfloor + 4 = \begin{cases} n+1 & \text{if n is odd} \\ n+2 & \text{if n is even} \end{cases} \tag{3.2}$$

This is a contradiction, which proves Claim i).

To prove Claims ii) and iii) observe that the convex hull of each halving pair $(e_{i,j}, e_{j+1,k})$ defines a quadrilateral $(i, j, j+1, k)$, see Figure 2. The halving edge witnessing the halving pair is contained in the corresponding convex hull: it is either the edge $e_{i,k}$, or one of the diagonals of the quadrilateral.

It is easy to see, that if either $e_{i,k}$ or one of the diagonals is intersected by an edge f, then f also intersects at least one edge of the pair $(e_{i,j}, e_{j+1,k})$.

Using this observation we prove the remaining two cases by contradiction: Assume there exists a halving edge and a halving pair which do not intersect, or two halving pairs which do not intersect. Then their corresponding halving edges (witnesses) do not intersect either, because they are contained in the quadrilaterals. This contradicts Claim i), and thus proves Claim ii) and iii). □

Definition 3.6. Let G be a complete convex geometric graph of even order n. We call an edge $e_{i,j}$ an *almost-halving edge* if $e_{i,j+1}$ is a halving edge.

Please observe that this definition and the following lemma are only stated (and valid) for even n; therefore if $e_{i,j}$ is an almost-halving edge, $e_{i-1,j}$ is a halving edge.

Lemma 3.7. *Let G be a complete convex geometric graph of even order n. Let f be an almost-halving edge, e a halving edge, and E a halving pair. i) f and e intersect, ii) f and E intersect.*

Proof. We prove Claim i) by contradiction. If e and f do not intersect, then they divide the set of vertices of G into three sets: one of size at least $\frac{n-2}{2}$, one of size at least $\frac{n-2}{2} - 1$, and one of size at least 4. In total the number of vertices is (at least):

$$2\left(\frac{n-2}{2}\right) - 1 + 4 = n + 1 \tag{3.3}$$

This is a contradiction, which proves Claim i). To prove Claim ii) we use Claim i): the halving edge witnessing E must intersect f. On the other hand such a halving edge is inside the convex hull of E, see Figure 2. From these two observations it follows that E and f intersect. □

We need two more concepts from the literature. A *thrackle* [21] is a graph drawn in the plane so that its edges are represented by Jordan arcs and any two distinct arcs either meet at exactly one common vertex or they cross (at exactly one point interior to both arcs). A *straight-line thrackle* is a thrackle in which all edges are straight-line segments. We say that a straight-line thrackle is *maximal*

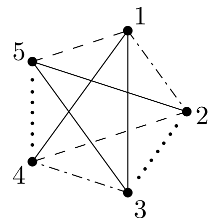


Figure 3: Proof of Theorem 3.8 for $n = 5$: $\alpha_1(\mathsf{K}_5) = 7$. Edges $e_{1,3}, e_{3,5}, e_{4,1}, e_{5,2}$ (solid) are colored with colors 1 to 4, respectively. Edges $e_{1,2}, e_{3,4}$ are colored with color 5; and edges $e_{2,3}, e_{4,5}$ are colored with color 6. Finally, edges $e_{2,4}, e_{1,5}$ are colored with color 7. Each pair of chromatic classes intersect, and each pair of edges with the same color are disjoint.

if it is not a proper subgraph of any other thrackle. Theorem 3.1 implies that the size of any straight-line thrackle is at most its number of vertices. In the following we always refer to a straight-line thrackle as thrackle, since we are only working with geometric embeddings of graphs.

Given a set $J \subseteq \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$, a *circulant graph* $C_n(J)$ of G is defined as the graph with vertex set equal to $V(G)$[1] and $E(C_n(J)) = \{e_{i,j} \in E(G)\colon j - i \equiv k \mod n, \text{ or } j - i \equiv -k \mod n, k \in J\}$. It is not hard to observe that the complete graph K_n is the circulant graph $C_n(J)$ when $J = \{1, ..., \lfloor \frac{n}{2} \rfloor\}$ (see, for example, [5, 16]). The natural partition of the edges of K_n induced by J is $E(K_n) = \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} E(C_n(i))$. We will consider this partition in the proof of Theorem 3.8. (Note that a partition is collectively exhaustive and mutually exclusive by definition). In this paper we use this concept for geometric graphs, in the natural way; see Figure 4 (left) for an example of a circulant geometric graph $C_n(J)$ with $J = \{\lfloor \frac{n}{2} \rfloor - 1\}$ and $n = 13$.

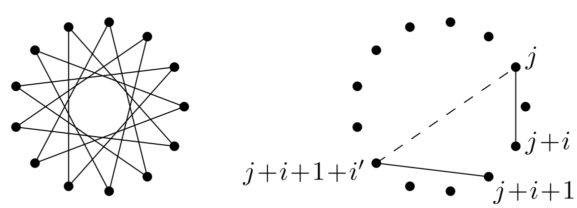


Figure 4: Examples for $n = 13$. Left: A circulant graph $C_n\left(\left\{\lfloor \frac{n}{2} \rfloor - 1\right\}\right)$. Right: A pair of edges (solid) with same color from $E(C_n(\{i, i'\}))$, with $i = 2$ and some fixed j. The witness of the halving pair is shown dashed.

Now we provide a lower bound on the achromatic index. We treat the case

[1]This definition is different from that usually given, in which the vertex set of the graph is $\mathbb{Z}_n$. However, as it is not hard to see that this definition is equivalent to the usual one, and to keep our arguments as simple as possible, we opt for this choice.

$n = 4$ first, because K_4 is the only complete convex geometric graph for which α_1 and ψ_1 are different. We prove that $\alpha_1(\mathsf{K}_4) = 4$ and $\psi_1(\mathsf{K}_4) = 5$. By Theorem 3.3 we have that $\psi_1(\mathsf{K}_4) \leq 5$, and by Figure 5 (left) we can conclude that $\psi_1(\mathsf{K}_4) = 5$. Now, by Figure 5 (right) we have $\alpha_1(\mathsf{K}_4) \geq 4$. Suppose that $\alpha_1(\mathsf{K}_4) = 5$, then in the coloring of K_4 with 5 colors, there must be four color classes of size one, and one color class of size two. The class of size two must be composed of two opposite edges of K_4. This implies that the remaining two opposite edges belong to classes of size one. But this is a contradiction because the coloring must be complete. Therefore, it follows that $\alpha_1(\mathsf{K}_4) = 4$.

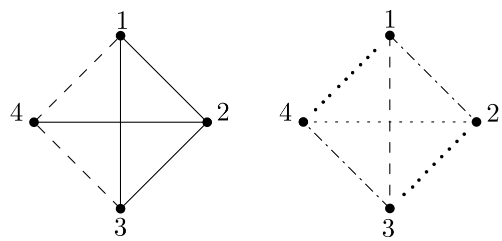


Figure 5: Left: $\psi_1(\mathsf{K}_4) = 5$. One color class of size two $\{e_{3,4}, e_{4,1}\}$; and four color classes of size one $\{e_{1,2}\}$,$\{e_{2,3}\}$,$\{e_{1,3}\}$,$\{e_{2,4}\}$. Right: $\alpha_1(\mathsf{K}_4) = 4$. Two color classes of size two: $\{e_{1,2}, e_{3,4}\}$, $\{e_{2,3}, e_{4,1}\}$; and two color classes of size one $\{e_{1,3}\}$, $\{e_{4,2}\}$.

The following theorem provides a lower bound on the achromatic index for the remaining cases.

Theorem 3.8. *Let* G *be a complete convex geometric graph of order* $n \neq 4$. *The achromatic index of* G *satisfies the following bound:*

$$\alpha_1(\mathsf{G}) \geq \lfloor \tfrac{n^2+n}{4} \rfloor.$$

Proof. The theorem follows easily for $n \leq 3$; we prove the case $n = 5$ in Figure 3. For $n > 5$, consider the following partition of the set of edges of G:

$$E(\mathsf{G}) = E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \bigcup E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup_{i \in I} E(C_n(\{i, \lfloor \tfrac{n}{2} \rfloor - 1 - i\})) \quad (3.4)$$

where $I = \{1, \ldots, \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor\}$.

Observe that the first term is a circulant graph of halving edges and thus, by Lemma 3.5, its set of edges defines a thrackle. This thrackle is maximal (containing n edges) if n is odd but it is not maximal (containing only $\frac{n}{2}$ edges) if n is even.

Note further, that for fixed i the third term is either the union of two circulant graphs of size n, or one circulant graph of size n (only in the case when $i = \lfloor \frac{n}{2} \rfloor - 1 - i$).

If n is odd, then the edge set of G is partitioned into $\frac{n-1}{2}$ circulant graphs, each of them of size n. If n is even, then the edge set of G is partitioned into

$\frac{n}{2}-1$ circulant graphs, each of them of size n, plus one circulant graph of size $\frac{n}{2}$. Using partition 3.4, we give a coloring on the edges of G, and prove that this coloring is proper and complete.

We start by coloring all circulant graphs in the third term of the partition, except for $i=\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor$.

In the following we set $i'=\lfloor\frac{n}{2}\rfloor-1-i$ and therefore refer to $C_n(\{i,\lfloor\frac{n}{2}\rfloor-1-i\})$ as $C_n(\{i,i'\})$. For every $i\in I\setminus\left\{\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor\right\}$ we assign colors to $C_n(\{i,i'\})$ using the following function.

$$f_i\colon E(C_n(\{i,i'\}))\longrightarrow\{(i-1)n+1,\dots,(i-1)n+n\}\text{ such that:}$$

$$\begin{aligned}e_{j,j+i}&\mapsto (i-1)n+j,\text{ and}\\ e_{j+i+1,j+i+1+i'}&\mapsto (i-1)n+j.\end{aligned}$$

for $j\in\{1,\dots,n\}$. See Figure 4 (right) for an example with $i=2$.

The first rule colors the edges of $C_n(\{i\})$, while the second rule colors the edges of $C_n(\{i'\})$. For fixed i and j both rules assign the same color. Therefore, the chromatic classes are pairs of edges, one edge ($e_{j,j+i}$) from $C_n(\{i\})$ and one edge ($e_{j+i+1,j+i+1+i'}$) from $C_n(\{i'\})$. Observe that all these pairs are halving pairs ($e_{j,j+i},e_{j+i+1,j+i+1+i'}$) of G, because the edge $e_{j,j+i+1+i'}=e_{j,j+\lfloor\frac{n}{2}\rfloor}$ is halving.

Hence, the partial coloring so far is complete (by Lemma 3.5) and proper (because the two edges in each color class do not intersect).

The number of colors we have used up to this point is $N_1=n\left(\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor-1\right)$.

So far, we have only colored a subset of edges of the third term of the partition 3.4. This leaves the following parts uncolored:

$$E(C_n(\{\lfloor\tfrac{n}{2}\rfloor\}))\bigcup E(C_n(\{\lfloor\tfrac{n}{2}\rfloor-1\}))\bigcup E(C_n(\{i,i'\}))$$

where $i=\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor$ and $i'=\lfloor\frac{n}{2}\rfloor-1-i$.

These remaining circulant graphs are different if n is even or if n is odd. Further, we need to distinguish between the two cases $i=i'$ and $i\neq i'$ (for the remainder of the third term). This basically results in the four cases $n\equiv x \mod 4$, for $x\in\{0,1,2,3\}$.

In a nutshell, to color the remaining edges, first we color the thrackle, $E(C_n(\{\lfloor\frac{n}{2}\rfloor\}))$ (if n is even together with one half of $E(C_n(\{\lfloor\frac{n}{2}\rfloor-1\}))$). Then we color (the remaining half of) the circulant graph $C_n(\{\lfloor\frac{n}{2}\rfloor-1\})$ together with $C_n(\{i,i'\})$ ($i=\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor$ and $i'=\lfloor\frac{n}{2}\rfloor-1-i$). In each step we will prove that the (partial) coloring is proper and complete.

1. Case $n>5$ is odd. To color the maximal thrackle, $E(C_n(\{\lfloor\frac{n}{2}\rfloor\}))$, we assign colors to its edges using the function

$$f\colon E(C_n(\{\lfloor\tfrac{n}{2}\rfloor\}))\longrightarrow\{N_1+1,\dots,N_1+n\}\text{ such that:}$$

$$e_{j,j+\lfloor \frac{n}{2} \rfloor} \mapsto N_1 + j,$$

for each $j \in \{1,\ldots,n\}$. Observe that $E(C_n(\{\lfloor \frac{n}{2} \rfloor\}))$ is a set of n halving edges. See Figure 6 (left) for an example of such a thrackle.

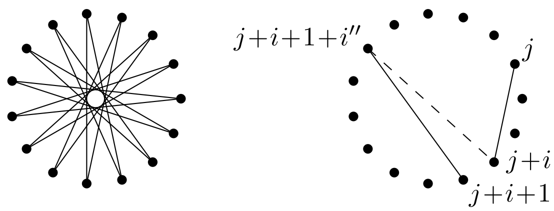


Figure 6: Examples for $n = 15$. Left: A circulant graph $C_n\left(\left\{\left\lfloor \frac{n}{2} \right\rfloor\right\}\right)$ of halving edges if n is odd. Right: Halving pair (solid) with color $N_2 + j$ from $E(C_n(\{i, i''\}))$, with $n \equiv 3 \mod 4$ and some fixed j. The witness of the halving pair is shown dashed.

The coloring so far is proper, because each new chromatic class has size one. Further, each chromatic class so far consists of either a halving edge or a halving pair. Hence, by Lemma 3.5, the coloring is also complete.

It is easy to see that we are using $N_2 = N_1 + n = n\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ colors so far.

The remaining uncolored edges are:

$$E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$. We will color these two circulant graphs together. Let $i'' = \lfloor \frac{n}{2} \rfloor - 1$. As n is odd, $C_n(\{i''\})$ consists of n edges. The size of $E(C_n(\{i, i'\}))$ depends on the two cases $i = i'$ and $i \neq i'$.

(a) $i = i'$: As n is odd, $n \equiv 3 \mod 4$. The circulant graph $C_n(\{i, i'\}) = C_n(\{i\})$ is of size n. Thus, $2n$ edges remain uncolored.

We assign n colors to the $2n$ edges of $C_n(\{i, i''\})$ as follows:

$$f_i \colon E(C_n(\{i, i''\})) \longrightarrow \{N_2 + 1, \ldots, N_2 + n\}, \text{ such that}$$

$$\begin{aligned} e_{j,j+i} &\mapsto N_2 + j, \\ e_{j+i+1,j+i+1+i''} &\mapsto N_2 + j \end{aligned}$$

for $j \in \{1, \ldots, n\}$.

Each new chromatic class consists of a pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i''})$ of edges. See Figure 6 (right) for an example of such a pair. Because the edge $e_{j+i,j+i+1+i''} = e_{j+i,j+i+\lfloor \frac{n}{2} \rfloor}$ is a halving edge, the pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i''})$ is a halving pair. Therefore, all edges are colored and each chromatic class consists of either a halving edge or a halving pair. By Lemma 3.5 the coloring is complete and proper (as the edges of halving pairs are disjoint).

The total number of colors used is $N_3 = N_2 + n = n(\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor + 1)$, that is $N_3 = \left\lfloor \frac{n^2+n}{4} \right\rfloor$ colors, as $n \equiv 3 \mod 4$ in this case.

(b) $i \neq i'$: As n is odd, $n \equiv 1 \mod 4$. The circulant graph $C_n(\{i, i'\})$ is of size $2n$. Thus, $3n$ edges remain uncolored. We assign n colors to the $2n$ edges of $C_n(\{i, i'\})$ and $\lfloor \frac{n}{2} \rfloor$ colors to the n edges of $C_n(\{i''\})$ as follows:

$$f_i\colon E(C_n(\{i, i', i''\})) \longrightarrow \{N_2 + 1, \dots, N_2 + n + \lfloor \tfrac{n}{2} \rfloor\}, \text{ such that}$$

$$\begin{aligned} e_{j,j+i} &\mapsto N_2 + j, \\ e_{j+i+1,j+i+1+i'} &\mapsto N_2 + j, \end{aligned}$$

for $j \in \{1, \dots, n\}$, and

$$\begin{aligned} e_{j,j+i''} &\mapsto N_2 + n + j, \\ e_{j+i''+1,j+i''+1+i''} &\mapsto N_2 + n + j, \end{aligned}$$

for $j \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. See Figure 7 (left and middle) for examples.

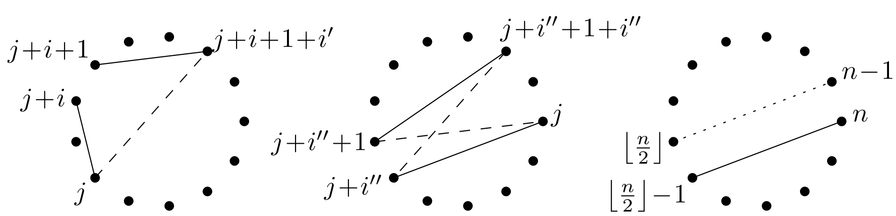


Figure 7: Examples with $n = 13$, for n is odd and $i \neq i'$: $n \equiv 1 \mod 4$. Left: Halving pair with color $N_2 + j$ from $E(C_n(\{i, i'\}))$. Middle: Halving pair with color $N_2 + n + j$ from $E(C_n(\{i''\}))$. Both for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed. Right: The single remaining edge $e_{n,\lfloor \frac{n}{2} \rfloor - 1}$ (solid) is combined with the halving edge $e_{\lfloor \frac{n}{2} \rfloor, n-1}$ (dotted), colored with color $N_1 + \lfloor \frac{n}{2} \rfloor$.

Each new chromatic class consists of a pair of edges. These pairs are either $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ or $(e_{j,j+i''}, e_{j+i''+1,j+i''+1+i''})$ combined from the edges of $C_n(\{i, i'\})$ or $C_n(\{i''\})$, respectively. The pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ is a halving pair with the halving edge $e_{j,j+i+1+i'} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ as witness, and $(e_{j,j+i''}, e_{j+i''+1,j+i''+1+i''})$ is a halving pair with the halving edges $e_{j,j+i''+1} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ and $e_{j+i'',j+i''+1+i''} = e_{j+i'',j+i''+\lfloor \frac{n}{2} \rfloor}$ as witnesses. Each chromatic class so far consists of either a halving edge or a halving pair. Hence, the coloring is complete (by Lemma 3.5) and proper (as the edges of halving pairs are disjoint).

Note, that a single edge, $e_{n,\lfloor\frac{n}{2}\rfloor-1}$ of $C_n(\{i''\})$, remains uncolored. We add this edge to the chromatic class (with color $N_1+\lfloor\frac{n}{2}\rfloor$) containing the halving edge $e_{\lfloor\frac{n}{2}\rfloor,n-1} = e_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}$. See Figure 7 (right). Observe, that $e_{n,\lfloor\frac{n}{2}\rfloor-1}$ and $e_{\lfloor\frac{n}{2}\rfloor,n-1}$ are disjoint, thus the coloring remains proper. Further, adding an edge to an existing chromatic class of a complete coloring, maintains the completeness of the coloring.

As all edges are colored, the total number of colors used is $N_3 = N_2+n+\lfloor\frac{n}{2}\rfloor = n(\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor+1)+\lfloor\frac{n}{2}\rfloor$, that is $N_3 = \left\lfloor\frac{n^2+n}{4}\right\rfloor$, as $n \equiv 1 \mod 4$ in this case.

2. Case $n > 5$ is even. Recall that only N_1 chromatic classes exist so far, each containing a halving pair of edges. The thrackle $E(C_n(\{\lfloor\frac{n}{2}\rfloor\})) = E(C_n(\{\frac{n}{2}\}))$ is not maximal in this case. See Figure 8 (left). To get a maximal thrackle we add half the edges of $C_n(\{\frac{n}{2}-1\})$ to $C_n(\{\frac{n}{2}\})$. Note that $E(C_n(\{\frac{n}{2}-1\}))$ is the set of almost-halving edges if n is even.

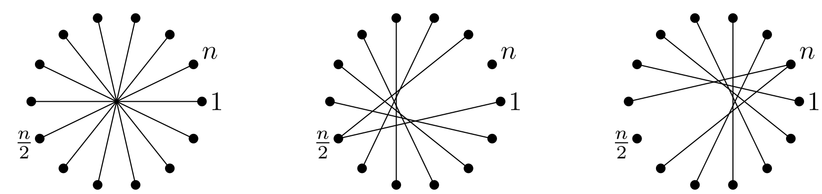


Figure 8: Examples with $n = 14$, for the case when n is even. Left: The thrackle, $E(C_n(\frac{n}{2}\}))$, of the $\frac{n}{2}$ halving edges. Middle: The thrackle, $E(C'_n(\{\frac{n}{2}-1\}))$, of the first $\frac{n}{2}$ almost-halving edges of $E(C_n(\{\frac{n}{2}-1\}))$. Right: The thrackle, $E(C''_n(\{\frac{n}{2}-1\}))$, of the second $\frac{n}{2}$ almost-halving edges of $E(C_n(\{\frac{n}{2}-1\}))$.

Let the thrackle $E(C'_n(\{\frac{n}{2}-1\})) = \{e_{1,\frac{n}{2}},\dots,e_{\frac{n}{2},n-1}\}$ and the thrackle $E(C''_n(\{\frac{n}{2}-1\})) = \{e_{\frac{n}{2}+1,n},\dots,e_{n,\frac{n}{2}-1}\}$ define the two halves of $C_n(\{\frac{n}{2}-1\})$ with $\frac{n}{2}$ almost-halving edges each. See Figure 8 (middle and right). It is easy to see that $E(C'_n(\{\frac{n}{2}-1\}))$ is a thrackle (all its edges intersect each other). Further, by Lemma 3.7, each almost-halving edge intersects each halving edge. Thus, $E(C_n(\{\frac{n}{2}\}) \cup C'_n(\{\frac{n}{2}-1\}))$ is a maximal thrackle of size n. The following function assigns one color to each edge of this maximal thrackle.

$$f\colon E(C_n(\{\frac{n}{2}\}) \cup C'_n(\{\frac{n}{2}-1\})) \longrightarrow \{N_1+1,\dots,N_1+n\}, \text{ such that}$$

$$\begin{aligned} e_{j,j+\frac{n}{2}} &\mapsto N_1+j, \\ e_{j,j+\frac{n}{2}-1} &\mapsto N_1+\frac{n}{2}+j \end{aligned}$$

for each $j \in \{1, \ldots, \frac{n}{2}\}$.

The coloring so far is proper, because each new chromatic class has size one. Further, each chromatic class consists of either a halving edge, a halving pair, or an almost-halving edge. The almost-halving edges used so far form a thrackle and thus intersect each other. Hence, by Lemmas 3.5 and 3.7, the coloring is also complete. It is easy to see that we are using $N_2 = N_1 + n = n\lfloor \frac{n-2}{4} \rfloor$ colors so far.

The remaining uncolored edges are

$$E(C''_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \frac{n-2}{4} \rfloor$ and $i' = \frac{n}{2} - 1 - i$ (as n is even). We will color these two circulant graphs together. For brevity, let $i'' = \frac{n}{2} - 1$ and let $C''_n(\{i''\})$ be the set of the remaining $\frac{n}{2}$ almost-halving edges. The size of $E(C_n(\{i, i'\}))$ depends on the two cases $i = i'$ and $i \neq i'$.

(a) $i = i'$: As n is even, $n \equiv 2 \mod 4$. The circulant graph $C_n(\{i, i'\}) = C_n(\{i\})$ is of size n. Thus, $n + \frac{n}{2}$ edges remain uncolored. We assign $\frac{n}{2} + \lfloor \frac{n}{4} \rfloor$ colors to the $n + \frac{n}{2}$ edges of $C_n(\{i\}) \cup C''_n(\{i''\})$ as follows:

$f_i \colon E(C_n(\{i\}) \cup C''_n(\{i''\})) \longrightarrow \{N_2 + 1, \ldots, N_2 + \frac{n}{2} + \lfloor \frac{n}{4} \rfloor\}$, such that

$$\begin{aligned} e_{\frac{n}{2}+j, \frac{n}{2}+j+i''} &\mapsto N_2 + j, \\ e_{\frac{n}{2}+j+i''+1, \frac{n}{2}+j+i''+1+i} &\mapsto N_2 + j \end{aligned}$$

for $j \in \{1, \ldots, \frac{n}{2}\}$, and

$$\begin{aligned} e_{\frac{n}{2}+j, \frac{n}{2}+j+i} &\mapsto N_2 + \tfrac{n}{2} + j, \\ e_{\frac{n}{2}+j+i+1, \frac{n}{2}+j+i+1+i} &\mapsto N_2 + \tfrac{n}{2} + j \end{aligned}$$

for $j \in \{1, \ldots, \lfloor \frac{n}{4} \rfloor\}$. See Figure 9 (left and middle) for examples.

Each new chromatic class consists of a halving pair of edges from $E(C_n(\{i\}) \cup C''_n(\{i''\}))$, either $(e_{\frac{n}{2}+j, \frac{n}{2}+j+i''}, e_{\frac{n}{2}+j+i''+1, \frac{n}{2}+j+i''+1+i})$ with the halving edge $e_{\frac{n}{2}+j, \frac{n}{2}+j+i''+1} = e_{\frac{n}{2}+j, \frac{n}{2}+j+\frac{n}{2}}$ as witness, or $(e_{\frac{n}{2}+j, \frac{n}{2}+j+i}, e_{\frac{n}{2}+j+i+1, \frac{n}{2}+j+i+1+i})$ with, again, the halving edge $e_{\frac{n}{2}+j, \frac{n}{2}+j+i+1+i} = e_{\frac{n}{2}+j, \frac{n}{2}+j+\frac{n}{2}}$ as witness. Each chromatic class so far consists of either a halving edge, a halving pair, or one of the $\frac{n}{2}$ almost-halving edges that form a thrackle. Hence, the coloring is complete (by Lemmas 3.5 and 3.7) and proper (as the edges of halving pairs are disjoint).

Note, that a single edge, $e_{\frac{n}{2}+\lfloor \frac{n}{4} \rfloor+1, n}$ of $C_n(\{i\})$, remains uncolored. We add this edge to the chromatic class (with color $N_1 + 1$) containing the halving edge $(e_{1, \frac{n}{2}+1})$. See Figure 9 (right). Observe, that $e_{\frac{n}{2}+\lfloor \frac{n}{4} \rfloor+1, n}$ and $(e_{1, \frac{n}{2}+1})$ are disjoint. Thus, the coloring remains

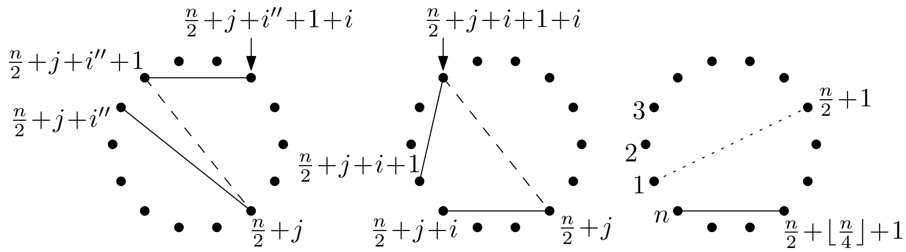


Figure 9: Examples with $n = 14$, for the case when n is even and $i = i'$: $n \equiv 2 \mod 4$. Left: Halving pair with color $N_2 + j$. Middle: Halving pair with color $N_2 + \frac{n}{2} + j$. Both for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed. Right: The single remaining edge $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ (solid) is combined with the halving edge $(e_{1,\frac{n}{2}+1})$ (dotted), colored with color $N_1 + 1$.

proper. Further, adding an edge to an existing chromatic class of a complete coloring maintains the completeness of the coloring.

As all edges are colored, the total number of colors used is $N_3 = N_2 + \frac{n}{2} + \lfloor\frac{n}{4}\rfloor = n\lfloor\frac{n-2}{4}\rfloor + \frac{n}{2} + \lfloor\frac{n}{4}\rfloor$, that is $N_3 = \left\lfloor\frac{n^2+n}{4}\right\rfloor$, as $n \equiv 2 \mod 4$ in this case.

(b) $i \neq i'$: As n is even, $n \equiv 0 \mod 4$. The circulant graph $C_n(\{i, i'\})$ is of size $2n$. Thus, $2n + \frac{n}{2}$ edges remain uncolored. We assign $\frac{n}{2} + 3\frac{n}{4}$ colors to the $2n + \frac{n}{2}$ edges of $C_n(\{i, i'\}) \cup C''_n(\{i''\})$ as follows:

$f_i \colon E(C_n(\{i, i'\}) \cup C''_n(\{i''\})) \longrightarrow \{N_2+1, \ldots, N_2+\frac{n}{2}+3\frac{n}{4}\}$, such that

$$\begin{aligned} e_{\frac{n}{2}+j,\frac{n}{2}+j+i''} &\mapsto N_2 + j, \\ e_{\frac{n}{2}+j+i''+1,\frac{n}{2}+j+i''+1+i} &\mapsto N_2 + j, \end{aligned}$$

$$\begin{aligned} e_{3\frac{n}{4}+j,3\frac{n}{4}+j+i''} &\mapsto N_2 + \tfrac{n}{4} + j, \\ e_{3\frac{n}{4}+j+i''+1,3\frac{n}{4}+j+i''+1+i'} &\mapsto N_2 + \tfrac{n}{4} + j, \end{aligned}$$

for each $j \in \{1, \ldots, \frac{n}{4}\}$, and

$$\begin{aligned} e_{\frac{n}{4}+j,\frac{n}{4}+j+i} &\mapsto N_2 + \tfrac{n}{2} + j, \\ e_{\frac{n}{4}+j+i+1,\frac{n}{4}+j+i+1+i'} &\mapsto N_2 + \tfrac{n}{2} + j \end{aligned}$$

for each $j \in \{1, \ldots, 3\frac{n}{4}\}$. See Figure 10 for an example of these three different types of pairs of edges.

Each new chromatic class consists of a halving pair of edges from $C_n(\{i, i'\}) \cup C''_n(\{i''\})$, either $(e_{\frac{n}{2}+j,\frac{n}{2}+j+i''}, e_{\frac{n}{2}+j+i''+1,\frac{n}{2}+j+i''+1+i})$

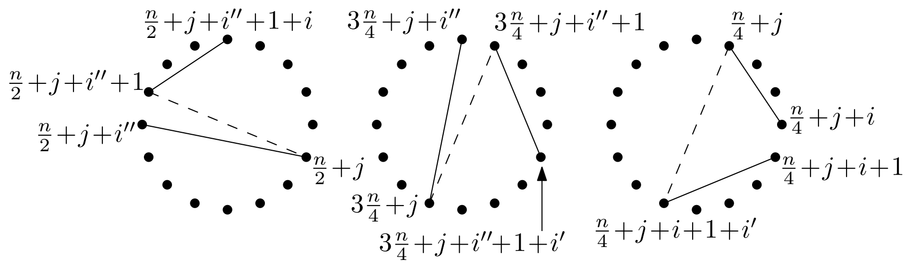


Figure 10: Examples with $n = 16$, for the case when n is even and $i \neq i'$: $n \equiv 0 \mod 4$. Left: Halving pair with color $N_2 + j$. Middle: Halving pair with color $N_2 + \frac{n}{4} + j$. Right: Halving pair with color $N_2 + \frac{n}{2} + j$. All for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed.

with the halving edge $e_{\frac{n}{2}+j,\frac{n}{2}+j+i''+1} = e_{\frac{n}{2}+j,\frac{n}{2}+j+\frac{n}{2}}$ as witness (Figure 10 (left)), $(e_{3\frac{n}{4}+j,3\frac{n}{4}+j+i''}, e_{3\frac{n}{4}+j+i''+1,3\frac{n}{4}+j+i''+1+i'})$ with the halving edge $e_{3\frac{n}{4}+j,3\frac{n}{4}+j+i''+1} = e_{3\frac{n}{4}+j,3\frac{n}{4}+j+\frac{n}{2}}$ as witness (Figure 10 (middle)), or $(e_{\frac{n}{4}+j,\frac{n}{4}+j+i}, e_{\frac{n}{4}+j+i+1,\frac{n}{4}+j+i+1+i'})$ with, again, the halving edge $e_{\frac{n}{4}+j,\frac{n}{4}+j+i+1+i'} = e_{\frac{n}{4}+j,\frac{n}{4}+j+\frac{n}{2}}$ as witness (Figure 10 (right)). Each chromatic class so far consists of either a halving edge, a halving pair, or one of the $\frac{n}{2}$ almost-halving edges that form a thrackle. Hence, the coloring is complete (by Lemmas 3.5 and 3.7) and proper (as the edges of halving pairs are disjoint).

As all edges are colored, the total number of colors used is $N_3 = N_2 + \frac{n}{2} + 3\frac{n}{4} = n\lfloor\frac{n-2}{4}\rfloor + \frac{n}{2} + 3\frac{n}{4}$, that is $N_3 = \left\lfloor\frac{n^2+n}{4}\right\rfloor$, as $n \equiv 0 \mod 4$ in this case.

□

Proof of Theorem 2.1 i). For $n \neq 4$, by Theorem 3.8 we get that $\left\lfloor\frac{n^2+n}{4}\right\rfloor \leq \alpha_1(\mathsf{G})$, and by Theorem 3.3 and Equation 2.4 we conclude that $\alpha_1(\mathsf{G}) = \psi_1(\mathsf{G}) = \left\lfloor\frac{n^2+n}{4}\right\rfloor$.

□

4 On $\psi_g(K_n)$

In this section, we consider point sets in general position in the plane and present lower and upper bounds for the geometric pseudoachromatic index. Recall that the geometric pseudoachromatic index of a graph G is defined as:

$$\psi_g(G) = \min\{\psi_1(\mathsf{G})\colon \mathsf{G} \text{ is a geometric graph of } G\}.$$

To achieve the upper bound of $\psi_g(K_n) \leq 0.1781n^2 + \Theta(n)$, we count the total number of edge intersections. That is, we count the number of edges that have a common endpoint and the number of edges that cross, and then we add these two quantities together. It is clear that $\binom{\psi_1(\mathsf{G})}{2}$ must be less than the total number of edge intersections. To achieve the lower bound we exhibit a complete coloring for any geometric graph using $0.0710n^2 - \Theta(n)$ colors.

4.1 Upper bound for $\psi_g(K_n)$

Let $G = (V, E)$ be a graph, and let G be a geometric representation of G. Consider two intersecting edges in G, the intersection might occur either at a common interior point (crossing), or at a common end point (at a vertex).

On one hand, if we consider all edges of G and denote by m the total number of intersections that occur at vertices, m is precisely the number of edges in the line graph of G. That is, if $deg(v)$ is the degree of $v \in V$, then $m = \sum_{v \in V} \binom{deg(v)}{2}$.

On the other hand, the *rectilinear crossing number of* G, denoted by $\overline{cr}(\mathsf{G})$, is defined as the number of edge crossings that occur in G. Given a graph G, the *rectilinear crossing number of* G is the minimum number of crossings over all possible geometric embeddings of G; notationally

$$\overline{cr}(G) = \min\{\overline{cr}(\mathsf{G})\colon \mathsf{G} \text{ is a geometric graph of } G\}.$$

It seems natural that there should be a relationship between the rectilinear crossing number of a graph, and its geometric achromatic and pseudoachromatic indices. In the following, we establish bounds for $\psi_g(G)$ as a function of m and $\overline{cr}(G)$.

Lemma 4.1. *Let G be a geometric graph of order n, denote by $\overline{cr}(\mathsf{G})$ the number of edge crossings in G, and by m the total number of edge intersections occurring at vertices of G. Then:*

$$\psi_1(\mathsf{G}) \leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G}))}}{2} \right\rfloor.$$

Proof. The total number of edge intersections is $m + \overline{cr}(\mathsf{G})$. Then, $m + \overline{cr}(\mathsf{G}) \geq \binom{\psi_1(\mathsf{G})}{2}$ so that $\psi_1(\mathsf{G})(\psi_1(\mathsf{G}) - 1) \leq 2(m + \overline{cr(\mathsf{G})})$. Solving this inequality we get $\psi_1(\mathsf{G}) \leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G}))}}{2} \right\rfloor$. □

Using the above Lemma, we can establish the following result.

Theorem 4.2. *Let $G = (V, E)$ be a graph of order n, with $m = \sum_{v \in V} \binom{deg(v)}{2}$. Denote by $\overline{cr}(G)$ its rectilinear crossing number. Then*

$$\psi_g(G) \leq \left\lfloor \frac{1 + \sqrt{1 + 8(m + \overline{cr}(G)}}{2} \right\rfloor.$$

Proof. Let G_0 be a geometric representation of G such that $\overline{cr}(\mathsf{G}_0) = \overline{cr}(G)$; that is, G_0 is a geometric graph of G with minimum number of crossings. As a consequence of Lemma 4.1 we have the following:

$$\begin{aligned}\psi_g(G) &= \min\{\psi_1(\mathsf{G})\colon \mathsf{G} \text{ is a geometric graph of } G\} \leq \psi_1(\mathsf{G}_0)\\ &\leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G}_0))}}{2} \right\rfloor = \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(G))}}{2} \right\rfloor\end{aligned}$$

□

Establishing bounds for $\overline{cr}(K_n)$ is a well studied problem in the literature. If we use these results, we can give a better bound for $\psi_g(K_n)$. Note that in this case $m = n\binom{n-1}{2}$. The following result was shown in [1].

Theorem 4.3. $\overline{cr}(K_n) \leq c\binom{n}{4} + \Theta(n^3)$ *for* $c = 0.380488$.

Using this theorem we obtain:

Theorem 4.4. *Let K_n be the complete graph of order n. The geometric pseudoachromatic index of K_n has the following upper bound:*

$$\psi_g(K_n) \leq 0.1781n^2 + \Theta(n).$$

Proof. Since $m = \sum_{v\in V(K_n)} \binom{deg(v)}{2} = n\binom{n-1}{2}$, by Theorem 4.2 and 4.3,

$$\begin{aligned}\psi_g(K_n) &\leq \frac{1}{2}\sqrt{8\overline{cr}(K_n)) + \Theta(n^3)} + \Theta(1) = \frac{1}{2}\sqrt{8\frac{c}{4!}n^4 + \Theta(n^3)} + \Theta(1)\\ &= \sqrt{\frac{c}{12}}n^2 + \Theta(n) \leq 0.1781n^2 + \Theta(n).\end{aligned}$$

□

4.2 Lower bound for $\psi_g(K_n)$

In this section we present a lower bound for $\psi_g(K_n)$. First let us state a result which will be used later; this result was shown in [8].

Theorem 4.5. *Let S be a set of n points in general position in the plane. There are three concurrent lines that divide the plane into six parts each containing at least $\frac{n}{6} - 1$ points of S in its interior.*

Recall that a coloring is complete if each pair of colors appears on at least one pair of intersecting edges. In order to prove the lower bound for $\psi_g(K_n)$ stated in Theorem**??**, we exhibit a complete coloring with at least $0.0710n^2 - \Theta(n)$ color classes for any geometric graph K_n of K_n. For any K_n, let S be its set of $n = 13m + 6 + r$ points in general position in the plane, with $n > 18$ and $r < 13$. We will subdivide the set of edges of K_n into $12m^2$ pairwise disjoint subsets (subgraphs of K_n) and prove that assigning one color to each such subset results in a complete coloring. To this end, we first subdivide S into 7 disjoint subsets using a specific configuration $\mathcal{L} = \{\ell_1, \ldots, \ell_6\}$ of 6 lines as defined in the following. (See Figure 11 for a sketch of the configuration $\mathcal{L}$).

Let ℓ_1, ℓ_2 and ℓ_3 be 3 horizontal lines not containing any point of S, with A', B' being the resulting subsets of S of the points between ℓ_1 & ℓ_2, and ℓ_2 & ℓ_3, respectively, such that $|A'| = 12m + 6$ and $|B'| = m + r$. Let ℓ_4, ℓ_5, ℓ_6 be 3 concurrent lines that divide A' into 6 subsets, each containing at least $2m$ points. The existence of ℓ_4, ℓ_5, ℓ_6 is guaranteed by Theorem 4.5. Let p be the point of intersection of ℓ_4, ℓ_5, and ℓ_6. For each one of these 6 subsets of A', we choose a subset of exactly $2m$ points and denote them by A, B, C, D, E, F, listed in clockwise order. Further, let $G \subseteq B'$, such that $|G| = m$ (see Figure 11 again).

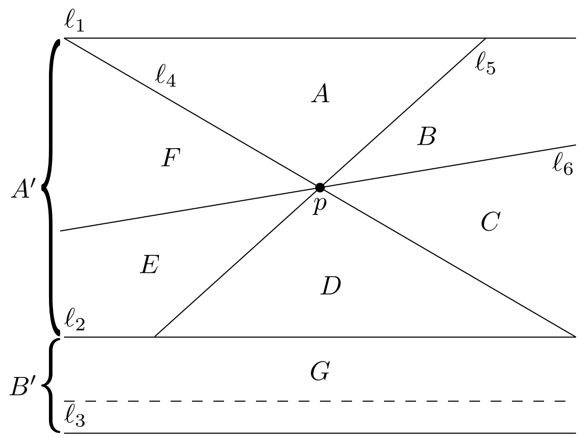


Figure 11: The line configuration $\mathcal{L}$.

Let $A = \{a_1, \ldots, a_{2m}\}, B = \{b_1, \ldots, b_{2m}\}, C = \{c_1, \ldots, c_{2m}\}, D = \{d_1, \ldots, d_{2m}\}, E = \{e_1, \ldots, e_{2m}\}, F = \{f_1, \ldots, f_{2m}\}$; and $G = \{g_1, \ldots, g_m\}$.

For $i, j \in \{1, \ldots, 2m\}$, consider the following sets of subgraphs of $\mathsf{K_n}$:

- The subgraphs $\mathsf{X}_{i,j}$ with vertex set $\{a_i, b_j, d_i, e_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{a_i b_j, b_j d_i, d_i e_j, e_j a_i\} \cup \begin{cases} \{a_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{d_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd} . \end{cases}$$

  Note that each $\mathsf{X}_{i,j}$ is a quadrilateral plus one edge. We call each quadrilateral $\mathsf{X}'_{i,j} \subseteq \mathsf{X}_{i,j}$, induced by vertices a_i, b_j, d_i, e_j.

- The subgraphs $\mathsf{Y}_{i,j}$ with vertex set $\{b_i, c_j, e_i, f_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{b_i c_j, c_j e_i, e_i f_j, f_j b_i\} \cup \begin{cases} \{b_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{e_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd} . \end{cases}$$

Let $\mathsf{Y}'_{i,j} \subseteq \mathsf{Y}_{i,j}$ be the quadrilateral induced by vertices b_i, c_j, e_i, f_j.

- The subgraphs $\mathsf{Z}_{i,j}$ with vertex set $\{c_i, d_j, f_i, a_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{c_i d_j, d_j f_i, f_i a_j, a_j c_i\} \cup \begin{cases} \{c_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{f_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd} . \end{cases}$$

Let $\mathsf{Z}'_{i,j} \subseteq \mathsf{Z}_{i,j}$ be the quadrilateral induced by vertices c_i, d_j, f_i, a_j.

Note that the set $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$, with $i, j \in \{1, \ldots, 2m\}$ contains exactly $12m^2$ edge-disjoint subgraphs. Recall that each subgraph $\mathsf{X}'_{i,j}, \mathsf{Y}'_{i,j}$ and $\mathsf{Z}'_{i,j}$, is a (not necessarily convex) quadrilateral. The following lemma shows that p is inside (the boundary of) each of these quadrilaterals.

Lemma 4.6. *Let p be the point of intersection of the three lines in Theorem 4.5. Then p is inside each of the polygons induced by the graphs $\mathsf{X}'_{i,j}, \mathsf{Y}'_{i,j}$, and $\mathsf{Z}'_{i,j}$, defined above.*

Proof. Consider the polygon P induced by $\mathsf{X}'_{i,j}$ and recall that the vertices of P are $\{a_i, b_j, d_i, e_j\}$. The line ℓ_4 separates the subsets A and B from the subsets D and E. Thus, ℓ_4 separates the edge $a_i b_j$ from the edge $d_i e_j$. The line ℓ_5 separates the subsets A and E from the subsets B and D. Thus, ℓ_5 intersects the edges $a_i b_j$ and $d_i e_j$, of P. Consider the segment of ℓ_5 defined by its intersection point with $a_i b_j$ and by its intersection point with $d_i e_j$; call this segment s. As ℓ_4 lies between $a_i b_j$ and $d_i e_j$, the point of intersection of ℓ_5 with ℓ_4 (which is the point p), lies in the interior of s. Furthermore, as ℓ_5 intersects P exactly twice, s is inside P and thus, p is inside P.

Analogously, p is inside the polygons induced by $\mathsf{Y}'_{i,j}$ and $\mathsf{Z}'_{i,j}$. □

Lemma 4.7. *For each pair of graphs from the set $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$ there exists a pair of edges, one of each graph, which intersect.*

Proof. We prove by contradiction. Assume that Q and R are two different graphs in $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$, such that no edge of Q intersects an edge of R. By Lemma 4.6, p is inside both polygons, P_Q and P_R, induced by Q and R, respectively. Since, by assumption, Q and R do not intersect, the boundaries of P_Q and P_R do not intersect either, and one polygon has to be contained inside the other (as both contain p). Without loss of generality, let P_Q be inside P_R. One edge of Q is connecting P_Q (in the interior of P_R) with a vertex in G (in the exterior of P_R) and therefore intersecting an edge of R. This is a contradiction to the assumption and the theorem follows. □

We can now conclude the proof of our main theorem.

Proof of Theorem 2.1 ii). For every geometric graph K_n of K_n with $n > 18$, one can construct the configuration $\mathcal{L}$. Using $\mathcal{L}$ we define the edge-disjoint subgraphs $\mathsf{G}_{i,j}$ ($\mathsf{G}_{i,j} \in \{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$). By construction, K_n contains $\frac{12}{169}n^2 - \Theta(n)$ of these subgraphs, and we assign a different color to each of them. By Lemma 4.7, each pair of these subgraphs intersects and therefore, $0.0710n^2 - \Theta(n) \leq \psi_g(K_n)$. (Note that for $\psi_g(K_n)$ the coloring needs to be complete, but not necessarily proper. Thus, uncolored edges, i.e., edges not in $\{\}$, can be added to any existing color class.) The upper bound $\psi_g(K_n) \leq 0.1781n^2 + \Theta(n)$ has been proven in Theorem**??**. □

5 Conclusions

In this paper we introduced the pseudoachromatic index and the achromatic index for geometric graphs. We proved the exact value of these parameters for complete geometric graphs defined by point sets in convex position, and gave upper and lower bounds for these parameters for (complete) geometric graphs defined by point sets in general position.

Some questions remain unanswered. Clearly, the main problem that remains open is to determine the exact values of the parameters for complete geometric graphs defined by point sets in general position. However, to shed some light on this issue, we might want to study the behavior of the parameters in complete geometric graphs defined by families of certain well known point sets, for instance, "double chain", "double circle" or "Horton sets". Further, it is clear that the number of crossings between edges is related to the values of the chromatic indices that we studied. An interesting question is to determine how these parameters are related with the crossing number of a graph. Finally, it is known that calculating the chromatic, achromatic number, and the pseudoachromatic number of a graph are NP-complete problems, see [7, 11, 22, 24, 25] and the references therein. Is it true that computing the geometric chromatic indices is also an NP-complete problem? Is it NP-complete to compute these indices if we consider some of the other adjacency criteria?

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