Guarding Rectangular Art Galleries

by

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Consider a rectangular art gallery divided into n rectangular rooms, such that any two rooms sharing a wall in common have a door connecting them. How many guards need to be stationed in the gallery so as to protect all of the rooms in our gallery? Notice that if a guard is stationed at a door, he will be able to guard two rooms. Our main aim in this paper is to show that $\lfloor n/2 \rfloor$ guards are always sufficient to protect all rooms in a rectangular art gallery. Extensions of our result are obtained for non-rectangular galleries and for 3-dimensional art galleries.

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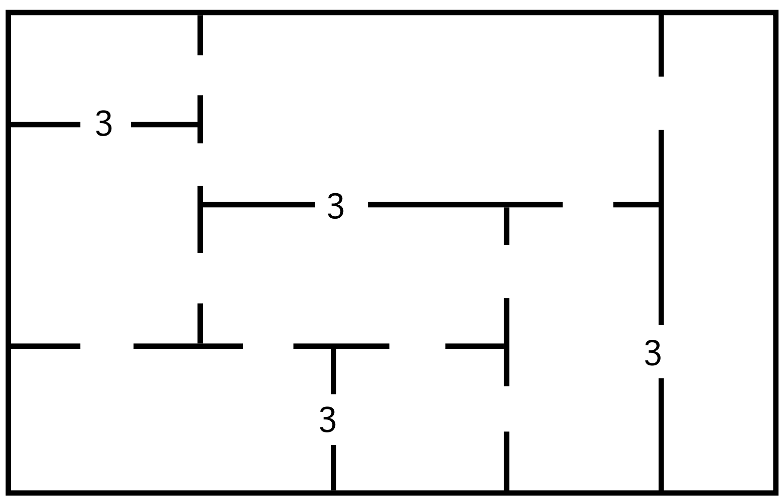
1. Introduction

An area of recent interest in Computational Geometry is that of Art Gallery Problems. A typical question of interest here is the following: suppose we have a set of art objects in an art gallery. How many guards are needed so that every art work is protected by at least one guard? In this paper we study the following problem.

Suppose we have a rectangular art gallery, subdivided into n rectangular rooms. Assume that any two adjacent rooms have a door connecting them. (See Figure 1.) How many guards need to be stationed in the gallery so as to protect all of the rooms in our gallery? Notice that if a guard is stationed at a door, he will be able to guard two rooms. Our main aim in this paper is to show that $\lfloor n/2 \rfloor$ guards are always sufficient to protect all rooms in a rectangular art gallery.



An example of an Art Gallery with eight rooms for which four guards suffice, showing the positions of their chairs.

Figure 1

In fact, we will prove an even stronger result; we prove that in an arbitrary art gallery (not necessarily convex, possibly having holes) with n rectangular rooms and k walls, $\lfloor (n+k/2) \rfloor$ guards are always sufficient and occasionally necessary to guard all the rooms in our gallery. As a consequence of our results, a linear time algorithm to find an optimal solution is obtained.

2. Rectangular galleries

An equivalent formulation to our first art gallery problem is the following. Suppose a rectangle **T** is decomposed into n rectangles, having mutually disjoint interiors. How few points in **T** are needed such that every subrectangle will contain at least one of the points? Trivially, n points will suffice. The example of n rectangles in a row shows that as many as $\lfloor (n/2) \rfloor$ points might be needed.

The purpose of this section is to show that $\lfloor (n/2) \rfloor$ points will always suffice, and to get a few extensions; more precisely, we state:

Theorem 1: If a rectangle **T** is decomposed into n rectangles, having mutually disjoint interiors, then there exists a set **S** consisting of at most $\lfloor (n/2) \rfloor$ points, such that every subrectangle meets **S**; moreover, all the points of **S** can be chosen so that each one of them belongs to at most two of the subrectangles.

Some definitions will be needed before we proceed with the proof of our result. A polygonal region is called **orthogonal** [5], if all its edges are parallel to either the x-axis or to the y-axis.

Given a decomposition of a rectangle **T** into n rectangles, we can obtain the dual graph $G(T)$ of the decomposition of **T** by representing each subrectangle of **T** by a vertex in $G(T)$, two vertices being adjacent if their corresponding rectangles share a line segment in their common boundary. See Figure 2.

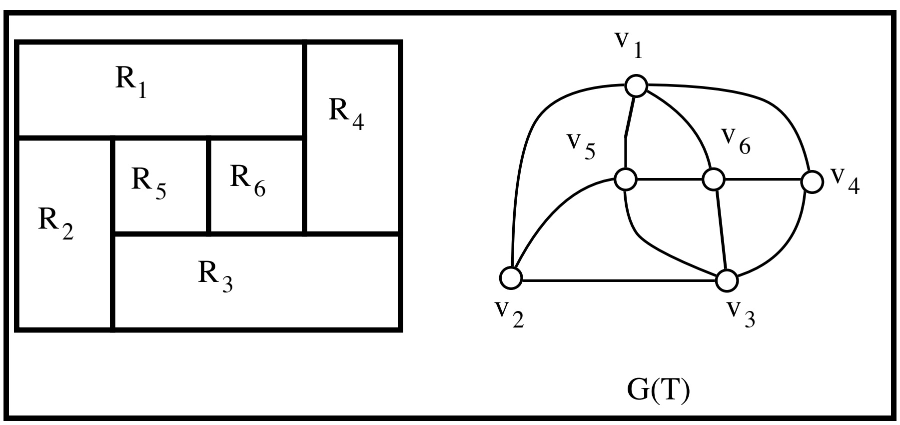


Figure 2

Notice that the union of the rectangles corresponding to the vertices of any connected subgraph of $G(T)$ form an orthogonal polygon. We are now ready to prove Theorem 1.

Proof of Theorem 1: Our result is based in the following property of G(T): if G(T) has an even number of nodes, then G(T) has a perfect matching M. Our n/2 points can now be chosen as follows: for every pair of elements $\{v_i,v_j\}$ in M choose a point in the common boundary of R_i and R_j. Clearly these points will cover all of the subrectangles of **T**. For instance for the graph shown in Figure 2, $\{v_1,v_5\}$, $\{v_2,v_3\}$ and $\{v_4,v_6\}$ form a perfect matching M for G(T). Three points can now be located one in the common boundaries of R_1,R_5 another in that of R_2,R_3 and the last one between R_4,R_6.

Let us now assume that G(T) has an even number of vertices. To prove that G(T) has a perfect matching, we proceed to prove that G(T) satisfies the conditions for the existence of a perfect matching stated in the well-known result of Tutte, namely, for any subset S of the vertices of G(T), the number of odd components of G(T)–S does not exceed |S|. Let k be the number of connected components of G(T)–S. Each connected component C_i of G(T)–S is represented by an orthogonal polygon P_i of **T**. Each such polygon has at least four corner points. The total number of corner points generated by the k components in G(T)–S is at least 4k. In other words, when we delete from **T** the rectangles representing vertices in S, we obtain a family of k disjoint orthogonal polygons contained in **T** with a total number of at least 4k corner points.

Our next observation is essential to our proof: when a rectangle represented by a point in S is now replaced, at most four corner points will disappear. (See Figure 3).

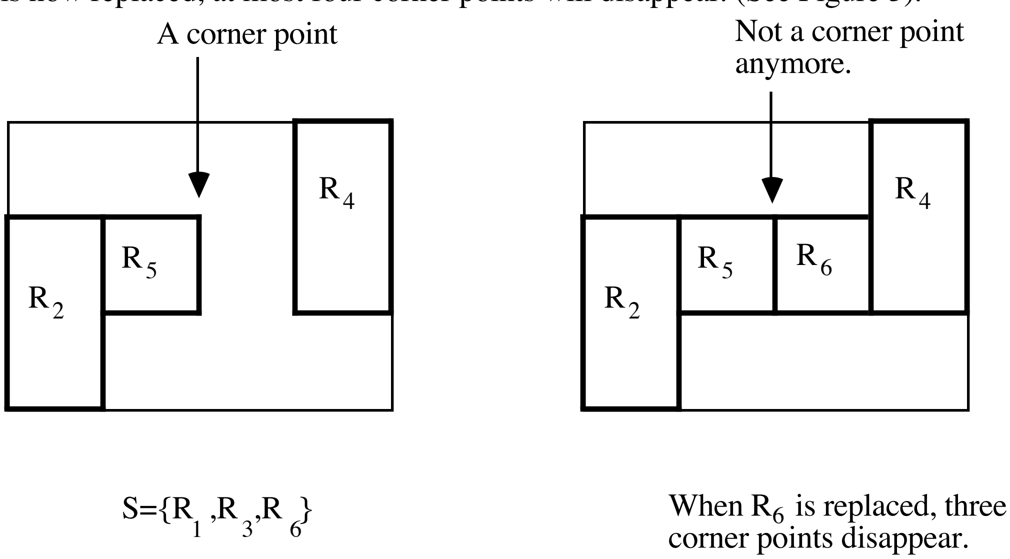


Figure 3

Once all rectangles in S are replaced, all the corner points generated by the components of G(T)–S will disappear, except for the four corner points of **T**; it follows that $k \leq |S|+1$.

The reader may verify that if k=|S|+1, then at least one of the components of G(T)–S is even, otherwise the number of vertices of G(T) would be odd, which contradicts the assumption that the number of vertices of G(T) is even.

For the case when n is odd, add an extra rectangle along one side of T and apply the previous arguments.

This completes the proof of Theorem 1.

Theorem 2: If a rectangle T is decomposed into n rectangles, having mutual disjoint interiors, then the dual graph of T has a Hamiltonian path.

Proof: Let a rectangle **T** be arbitrarily decomposed into n subrectangles. Let **T*** be the rectangle, obtained from the rectangle **T** by surrounding it with four additional subrectangles, as shown in Figure 4:

T* :

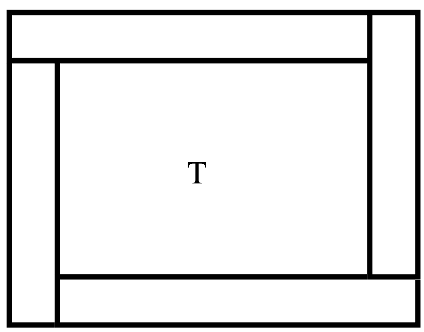


Figure 4

Main claim: The dual graph of the decomposition of **T***, in which the outer face is included, is 4-connected.

Proof of the claim: It suffices to show that the deletion of any three or fewer vertices from the dual graph of **T*** yields a connected subgraph. Equivalently, it suffices to show that if three or fewer subrectangles are to be avoided, then there is always a path from an interior point of any of the remaining subrectangles via the remaining subrectangles to the (remaining part of the) boundary of **T***. If the three subrectangles are all in the interior of **T***, they cannot block all the *four* directions to the boundary of **T***, available from any one of the remaining subrectangles; here a *direction* means up, down left or right. Similar arguments are used in case some of the three subrectangles are in the boundary of **T***. Therefore the dual graph of **T*** is 4-connected.

To complete our proof, let $\mathbf{T}^{\#}$ denote the decomposition of a rectangle into three copies of the decomposition T, joined by subrectangles, and surrounded by four rectangles, as shown in Figure 5.

$T^{\#}$:

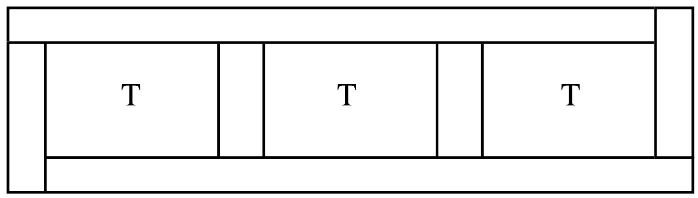


Figure 5

According to the previous part, the dual graph of $\mathbf{T}^{\#}$, where the outer face is included, is planar and 4-connected, hence it is Hamiltonian, by a theorem of Tutte [6]. It follows easily that every Hamiltonian circuit of $\mathbf{T}^{\#}$ must visit one of the copies of T in $\mathbf{T}^{\#}$ exactly once; thus the dual graph of T has a Hamiltonian path. This completes the proof of Theorem 2.

Our proof of Theorem 2 actually proves a slightly stronger statement, as follows.

Corollary 1: If a rectangle T is decomposed into rectangles, then the dual graph of T has a Hamiltonian path which starts at one side of T and ends up at the opposite side.

Theorem 2 implies Theorem 1, since every other edge on the Hamiltonian path yields a matching in the dual graph, having $\lfloor n/2 \rfloor$ edges. Algorithmically, finding the Hamiltonian path in the dual of T can be done by finding a Hamiltonian circuit in the dual of $\mathbf{T}^{\#}$; this can be done in linear time, using [1].

Next, we wish to show that if a rectangle F is decomposed into n subrectangles, and one tries to find a set S of points in a very naive way, so that every point of S meets at most two of the subrectangles, and every subrectangle meets S, then one never get more than [(3/4)n] points, and this bound is tight.

More precisely, let Algorithm A be the following one: given a rectangle F, decomposed into n subrectangles, choose a point that belongs to two rectangles which touch along a segment, delete these two subrectangles, and keep on choosing points on a pair of touching subrectangles which have not yet been deleted. Once this can no longer be done, the collection of subrectangles form an independent set in the dual graph. Then take one point in each one of these subrectangles.

Theorem 3: For every subdivision of a rectangle into n subrectangle, Algorithm A will always produce a set S that contains at most $\lfloor(3/4)n\rfloor$ points.

Proof of Theorem 3: Let a rectangle F be decomposed into n subrectangles, and suppose Algorithm A chooses x pairs of touching subrectangles. Then choose y points in the interior of the remaining subrectangles. It follows that $n = 2x + y$; however, the deletion of 2x subrectangles yields y connected components. Hence it follows, as in the proof of Theorem 1, that $2x \geq y - 1$; therefore at most n/2 of the rectangles were picked up by Algorithm A in singles, i.e. $n/2 \geq y$, implying that $\lfloor(3/4)n\rfloor \geq x + y$; as asserted.

To see that (3/4)n points might be assigned by Algorithm A, consider the following example, where the chosen points are marked.

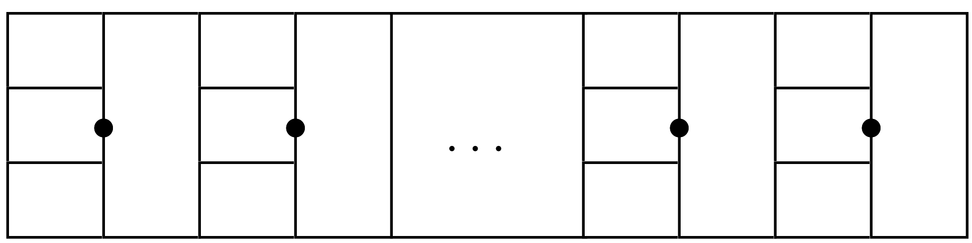


Figure 6

3. Non Rectangular Orthogonal Galleries

If our art gallery is not rectangular, our result is obviously false. In the worst case as many as n-1 points might be needed to meet all the subregions, as shown in Figure 7.

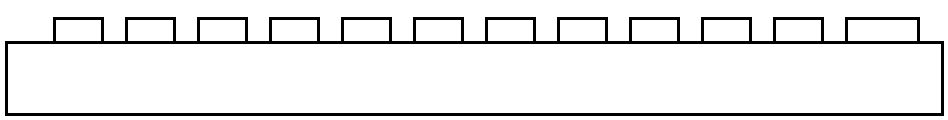


Figure 7

The following example shows a configuration in which about 2n/3 points are needed for the n rectangles:

Figure 8

For arbitrary orthogonal regions, we have the following.

Theorem 4: If **F** is an arbitrary orthogonal region in the plane, having k vertices, and if **F** is decomposed into n subrectangles having mutually disjoint interiors, then, for sufficiently large n, there exists a set **S** of at most $\lfloor(n+k)/2\rfloor$ points, so that every subrectangle meets **S**. Moreover, the points of **S** can be so chosen that each one of them meets at most two of the subrectangles.

Proof of Theorem 4: The proof is similar to the proof of Theorem 1, except that here F is wrapped with one layer consisting of k quadrangles; the rest of the proof is the same as in the proof of Theorem 2, hence it is omitted. In this case, too, finding S takes o((n+k)) time, by [1].

Recall that the *path number* p(**G**) of a graph **G** is defined as the minimum number of disjoint simple paths needed to cover all the vertices of **G**; thus **G** has a Hamiltonian path iff p(**G**) = 1.

We have another corollary.

Corollary 2: If an orthogonal region **F**, having k vertices, has a decomposition into n subrectangles, having mutually disjoint interiors, then the dual graph F^+ of the decomposition satisfies $k-1 \geq p(\mathbf{F}^+)$.

Proof: Let an orthogonal region F have k vertices; wrap it with k rectangles. The dual graph, including the outer face, is Hamiltonian; by deleting from the Hamiltonian circuit the k+1 points corresponding to the outer face and to the k added rectangles, we get at most k-1 paths.

Our next result is an extension of Theorem 2.

Theorem 5: For every orthogonal region **F**, possibly having holes, there exists a constant k, k = k(**F**), depending linearly on the number of vertices of F, such that if **F** is decomposed into n subrectangles, having mutually disjoint interiors, then there exists a set **S** consisting of $\lfloor(n+k)/2\rfloor$ points, such that every subrectangle meets **S**; moreover, the points of S can be chosen so that each one of them meets at most two of the subrectangles.

Proof of Theorem 5: Let F be an arbitrary orthogonal region, possibly having holes. Let B be the smallest rectangle which contains F; let the region inside B and outside F be decomposed by a *fixed* number k of rectangles. Every decomposition of F into n rectangles can be extended to a decomposition of the rectangle B into n+k rectangles; the rest follows from Theorem 1.

In a similar way, we have the following extension of Theorem 2.

Theorem 6: For every orthogonal region D in the plane, possibly having holes, there exists a constant k=k(D), depending linearly on the number of vertices of D, such that the dual graph of every decomposition of D into rectangles has a path number which is at most k.

4. General polygonal regions

In the general case, where a polygonal region is decomposed into n convex subregions, it is known [3] that about $\lfloor(2/3)n\rfloor$ points will be required; if the entire configuration is 3-connected, we have the following result.

Theorem 7: If **F** is an arbitrary polygonal region in the plane, having k vertices, and if **F** is decomposed into n convex subregions, then, for sufficiently large n, there exists a set **S** consisting of at most $\lfloor(2/3)(n+k)\rfloor$ points, such that every subregion meets **S**. Moreover, **S** can be so chosen that every point of **S** meets at most two of the subregions.

Proof of Theorem 7: Let **F** be an arbitrary polygonal region in the plane, having k vertices, and let **F** be decomposed into n convex subregions. Wrap **F** with a layer of quadrangles, as shown in Figure 9, and call the new decomposition **F*.**

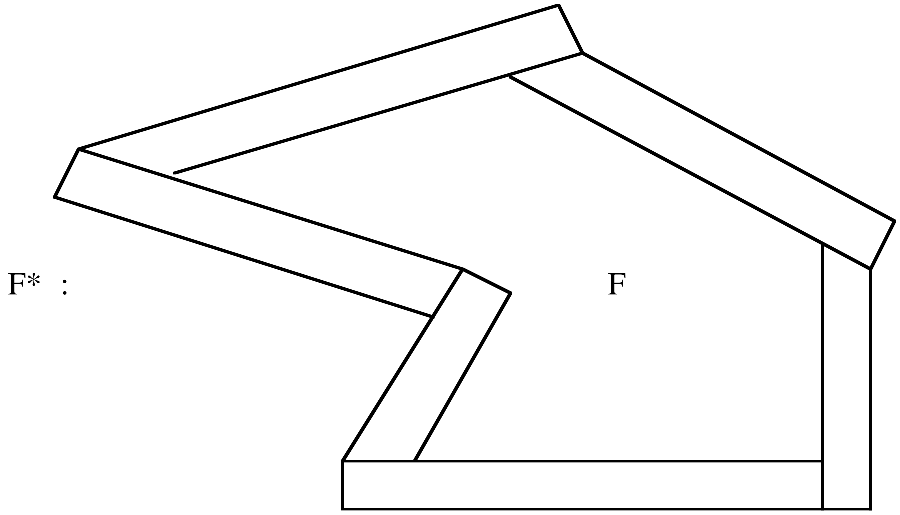


Figure 9

Claim: The dual graph of **F*** is 3-connected.

Proof: Let **A**, **B** and **C** be any three subregions of **F***, and let x be any point in the interior of **C**. Not all the rays from x to the part of bd(**F***) which is in **F* - (A $\cup$ B)** can be blocked by **A** $\cup$ **B**, since the projection of **A** (and **B**) on a small circle inside **C**, centred at x, is an arc which is less than half the circle, since x $\notin$ **A**. Thus, two such arcs add up to less than the full circle. Therefore the dual graph of **F* - (A $\cup$ B)** is connected. It follows that the dual graph of **F*** is 3-connected.

The dual graph of **F*** is, in addition, planar; herefore, by a theorem of Nishizeki [4], it has a matching which has $\lfloor(n+k)/3\rfloor$ edges. Thus, $\lfloor(n+k)/3\rfloor$ points of the set **S** can be chosen, in correspondence with the edges of this matching, lying (in the dual setting) on twice as many subregions. For the remaining one-third of the regions, take one point per region (on a proper boundary point of that subregion). This completes the proof of Theorem 7.

To see that about (2/3)n points are needed for a convex domain, which is decomposed into n subdomains, consider the dual graph of the m-th iterated Kleetope over the bi-pyramid, for large values of m (for details, see [2]).

Using the idea of the proof of Theorem 5, we get the following.

Theorem 8: For every polygonal region F in the plane, possibly having holes, there exists a constant k=k(F), such that for every decomposition of F into n convex subregions, there exists a set S consisting of at most $\lfloor(2/3)(n+k)\rfloor$ points, such that every subregion meets S; moreover, S can be so chosen that every point of S meets at most two of the subregions.

We remark that, unlike the existence of an upper bound for the path number of decompositions of orthogonal regions, as given in Corollary 2, there are no upper bounds to the corresponding path number for the decompositions of arbitrary polygonal region into convex subregions.

We conclude by showing that the situation in the 3-space is quite different from the one in the plane.

For every even integer n = 2k, consider the $n \times n \times n$ box $\{(x,y,z) \mid n \geq x,y,z \geq 0\}$, and decompose it into $2k^3 + 3k^2$ boxes as follows: let all the segments of the form [2i,2i+1] be denoted by **e**, let all the segments of the form [2i+1,2i] be denoted by **o**, and let the segment [0, 2k] be denoted by **K**. The decomposition consists of the $2k^3$ cubes of the form (**e**,**e**,**e**) and (**o**,**o**,**o**), and of the $3k^2$ long bricks of the form (**e**,**o**,**K**), (**K**,**e**,**o**) or (**o**,**K**,**e**). Figure 10.i shows the case k=3 in the i-th layer, i = odd, even; the clear squares are cross-sections of the bricks of the form (**e**,**o**,**K**).

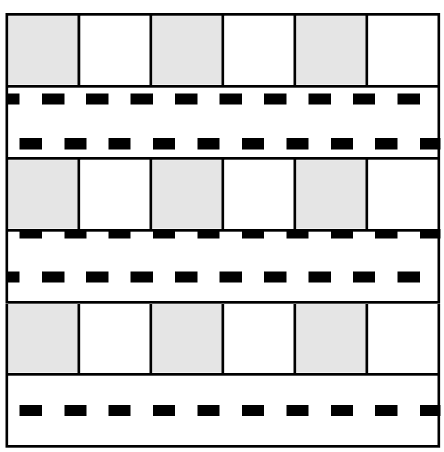


Figure 10.1

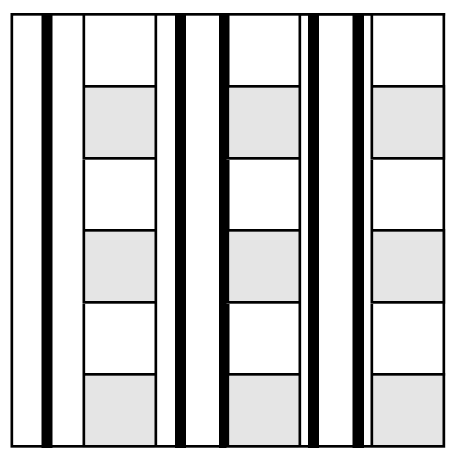


Figure 10.2

It is clear that the dual graph of this decomposition has $2k^3+3k^2$ vertices, and the deletion of the $3k^2$ of them yields an independent set of $2k^3$ vertices. Thus, if a set S of points has to meet all of the sub-boxes, then $|S| \geq 2k^3$, implying that

$$\lim |S|/(2k^3+3k^2) = 1.$$

Therefore, there exists no constant **c** < 1, for which *every* decomposition of a box into n sub-boxes has a set of **c**n points which meets all of the sub-boxes.

Moreover, since the deletion of $3k^2$ vertices from the dual graph in this example yields $2k^3$ components, it follows that the path number of the dual graph is at least $2k^3 - 3k^2$; hence there exist no upper bounds to the path number of the dual graph of decompositions of boxes into sub-boxes in the 3-space.

The argument in the proof of Theorem 1, of counting corners, is valid also in the case of decomposing a box into boxes, such that no two of them meet at just one point; thus, there is an analogue of Theorem 1 for this type of decomposition of boxes into boxes, in R^3.

References

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