Heavy non-crossing increasing paths and matchings on point sets

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Abstract. Let S be a permutation of $I_n = \{1, \dots, n\}$. The weight of a subsequence of S is the sum of its elements. We prove that any permutation S of I_n always contains an increasing or a decreasing subsequence of weight greater than $n\sqrt{n/3}$; our bound is asymptotically tight. We also show that S contains a *unimodal* subsequence of weight at least $n\sqrt{2n/3} - O(n)$. Our problem arises in the following geometric setting: Let P be a set of n points whose elements are labelled with the integers in I_n. A simple path of P is an increasing path if when we traverse it starting at one of its endpoints, the labels of its elements always increase. The weight of a path is the sum of the labels of its elements. We study the problem of finding simple increasing paths with large weight. We also study the problem of finding non-crossing matchings of P with large weight, where the weight of an edge with endpoints $i, j \in P$ is $\min\{i, j\}$, and the weight of a matching is the sum of the weights of its edges.

Introduction

Consider any permutation S of $I_n = \{1, \dots, n\}$. A well-known result of Erdős and Szekeres [3] asserts that any permutation of I_n always contains an increasing or a decreasing subsequence with at least $\lceil \sqrt{n} \rceil$ elements. The weight of a subsequence of S is the sum of its elements. If we consider the permutation $S = \{5, 2, 8, 1, 7, 4, 3, 6\}$, the weight of the increasing subsequence $\{2, 4, 6\}$ is equal to 12. Among all the increasing or decreasing subsequences of S, the one with maximum weight is $\{8, 7, 6\}$, with weight 21. In this paper we study the problem of finding the increasing or decreasing subsequence of a permutation with maximum weight. We prove that any permutation of I_n always contains an increasing or a decreasing subsequence with weight greater than $n\sqrt{n/3}$; our bound is asymptotically tight. The permutations obtained to solve this problem produce efficient packings of squares with areas $1, 2^2, \dots, n^2$. We also study the problem of finding unimodal subsequences of large weight of permutations of I_n. We show that any permutation of I_n always has a unimodal subsequence of weight at least $n\sqrt{2n/3} - O(n)$.

Our results are motivated by the following problem: Let P be a set of n points on the plane in general position such that its elements are labelled with the integers of I_n. Different elements of P receive different labels (we call this point set a labelled point set, or simply a point set). A path $\mathcal{W}$ whose vertices are elements of P is called *simple* if no two of its edges cross each other. $\mathcal{W}$ is called an *increasing path* if when we traverse it starting at one of its endpoints, the labels of its vertices always increase. The weight of a path is the sum of the labels of its vertices. Finding increasing or decreasing subsequences in permutations of I_n allows us to establish bounds on the weight of the heaviest simple increasing path in labelled point sets. For point sets in convex position, we use unimodal

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or anti-unimodal subsequences of large weight. We also study the problem of finding non-crossing matchings of labelled point sets with large weight, where the weight of the edge joining i and j is the smaller of $\{i, j\}$, and the weight of a matching is the sum of the weights of its edges. We show that a point set in convex position always has a matching of weight at least $n^2/5 - O(n)$. Point sets in general position always have matchings with weight at least $n^2/6 + \Omega(n)$.

Finding structures in point sets on the plane that optimize some given functions has been of interest to many computational geometers for some time. Problems studied so far include finding simple paths, matchings, cycles, and trees of maximum length; see Dumitrescu and Tóth [1]. Pach, Károlyi, and Tóth [5] show that if the edges of a complete geometric graph on $k^2 + 1$ points are colored red or blue, then there always exists a simple red or blue path of length $k + 1$. The problem of finding long simple increasing paths was first studied by Czyzowicz *et al.* [4]. They proved that any labelled point set in convex position contains a simple increasing path of length at least $\sqrt{2n}$. This bound was improved recently by Sakai and Urrutia [6] to $\sqrt{3n - 3/4} - 1/2$.

1 Heavy simple increasing paths

1.1 Heavy increasing subsequences of a permutation

Let $S = \{s(1), \ldots, s(n)\}$ be a permutation of I_n. To each $s(i)$ of S, we associate the point (x_i, y_i) as follows: x_i is the weight of the heaviest increasing subsequence of S ending at $s(i)$, and y_i is the weight of the heaviest decreasing subsequence of S starting at $s(i)$. If $S = \{4, 3, 7, 2, 5, 1, 6\}$, then we associate to $s(3) = 7$ the point $(x_3, y_3) = (4+7, 7+5+1) = (11, 13)$. We also associate to each $s(i)$ the square $SQ(i)$ whose top right vertex is (x_i, y_i) and whose bottom left vertex is the point $(x_i - s(i), y_i - s(i))$.

Observation If $i \neq j$, then $(x_i, y_i) \neq (x_j, y_j)$, and $SQ(i)$ and $SQ(j)$ have disjoint interiors.

Let α be the minimum value such that the square $\mathcal{SQ}$ with vertices $(0, 0), (0, \alpha), (\alpha, \alpha), (\alpha, 0)$ contains all $SQ(i)$. Since the area of $\mathcal{SQ}$ must be at least the total area of the $SQ(i)$, we must have $\alpha > n\sqrt{\frac{n}{3}}$. This implies:

Theorem 1.1. *Any permutation of I_n contains an increasing or a decreasing subsequence whose weight is greater than $n\sqrt{n/3}$. Our bound is asymptotically tight.*

To see that our bound is asymptotically tight, let $k = \sqrt[4]{4n^3/3}$, and $m = \sqrt{3n}/2$. Consider now the following permutation Π:

$$\lceil k \rceil, \lceil k \rceil - 1, \ldots, 1,\ \lceil \sqrt{2}k \rceil, \lceil \sqrt{2}k \rceil - 1, \ldots, \lceil k \rceil + 1,$$
$$\lceil \sqrt{3}k \rceil, \lceil \sqrt{3}k \rceil - 1, \ldots, \lceil \sqrt{2}k \rceil + 1,\ \ldots,\ n, n-1, \lceil \sqrt{m-1}\, k \rceil + 1.$$

Thus, Π consists of m blocks of decreasing integers such that, for each block, the sum of its elements is $n\sqrt{n/3} + O(n)$. On the other hand, the heaviest increasing subsequence of Π is the subsequence containing the elements $\lceil k \rceil, \lceil \sqrt{2}\, k \rceil, \lceil \sqrt{3}\, k \rceil, \ldots, n$, which again has weight $n\sqrt{n/3} + O(n)$. As a consequence, we have:

Theorem 1.2. *Any labelled point set with n elements has a simple increasing path of weight greater than $n\sqrt{n/3}$.*

1.2 Heavy increasing paths of point sets in convex position

A subsequence $\{s(i_1), \ldots, s(i_k)\}$ of a permutation S of I_n is called *unimodal* if there is a j, $1 \leq j \leq k$, such that $s(i_1) < \cdots < s(i_j) > \cdots > s(i_k)$; it is called *anti-unimodal* if for some j, $s(i_1) > \cdots > s(i_j) < \cdots < s(i_k)$. The next result is given without a proof.

Theorem 1.3. *Any permutation of I_n contains a unimodal subsequence of weight greater than $n\sqrt{2n/3} - O(n)$.*

Observe that the problem of finding a simple increasing path of maximum weight for a labelled point set in convex position can be reduced to that of finding a unimodal or anti-unimodal sequence of maximum weight in a permutation of I_n obtained from P by reading its elements starting at a suitable point of P. Thus the next result follows:

Theorem 1.4. *Any labelled point set in convex position has a simple increasing path of weight greater than $n\sqrt{2n/3} - O(n)$.*

The best upper bound we have for the weight of a unimodal or an anti-unimodal subsequence of a permutation is approximately $2n\sqrt{n/3}$, and is given by the permutation used in Theorem 1.1. On the other hand, the best upper bound we have for the weight of a simple increasing path of n labelled points in convex position is approximately $n\sqrt{2n}$. This is given by the following permutation Π' with $n = 2k^2$:

$$\begin{array}{lllll} n-k+1, & n-2k+1, & \ldots, & k+1, & 1, \\ n-k+2, & n-2k+2, & \ldots, & k+2, & 2, \\ & & \cdots & & \\ n=2k^2, & n-k, & \ldots, & 2k, & k. \end{array}$$

It is easy to see that the maximum weight unimodal or anti-unimodal subsequence of Π' has weight $\approx n\sqrt{2n}$; this weight is achieved by the subsequence

$$n-k+1, n-k+2, \ldots, n, n-k, \ldots, 2k, k.$$

2 Heavy non-crossing matchings

In this section, we study the problem of finding non-crossing matchings of a labelled point set that maximize the sum of the weights of its edges. We give lower bounds of the sums of the weights while efficient upper bounds are still open.

2.1 Point sets in convex position

Lemma 2.1. *The weight of any non-crossing perfect matching of a point set P in general position with $2m$ elements is at least $\binom{m+1}{2}$ and at most m^2. These bounds are tight.*

To prove the tightness of the lower bound, let P be an unlabelled point set with $2m$ elements in *convex* position. Color the elements of P red or blue in such a way that when we traverse the boundary of the convex hull of P the colors of its elements alternate. Observe that any edge of a perfect matching $\mathcal{M}$ of P joins a red and a blue point. Label the red and blue points of P with the integers $\{1, \ldots, m\}$ and $\{m+1, \ldots, 2m\}$, respectively, not necessarily in order. Then the weight of any edge of $\mathcal{M}$ belongs to $\{1, \ldots, m\}$. Since different edges in $\mathcal{M}$ have different weights, the weight of $\mathcal{M}$ is precisely $\binom{m+1}{2}$. The tightness of the upper bound is easy to prove.

Using similar arguments, we can prove:

Lemma 2.2. *Let P be a set of points in general position whose elements have been labelled with the set of integers $\{2r+1, 2r+2, \dots, 2m\}$. Then the weights of perfect matchings of P have the following bounds, and these bounds are tight:*

- *at least $(2r+1) + (2r+2) + \cdots + [2r + (m-r)] = 2r(m-r) + \binom{m-r+1}{2}$, and*
- *at most $(2r+1) + (2r+3) + \cdots + (2m-1) = 2r(m-r) + (m-r)^2$.*

Next we consider matchings that are not necessarily perfect.

Lemma 2.3. *Let P be a point set in convex position such that some of its elements are colored red and the rest blue. Then there is a non-crossing matching $\mathcal{M}$ of P that matches all, but at most two, elements of P such that the endpoints of each edge of $\mathcal{M}$ have the same color.*

We are now ready to prove the main result of this section.

Theorem 2.4. *Let P be a labelled point set in convex position. Then the heaviest non-crossing matching of P has weight at least $n^2/5 - O(n)$.*

Proof. To make our proof easy to understand, let us assume that P has $n = 10s$ elements. Discard from P all the elements with labels in $\{1, \dots, 2s\}$. Now color with blue and red all the elements of P with labels in $\{2s+1, \dots, 6s\}$ and $\{6s+1, \dots, 10s\}$, respectively. By Lemma 2.3, we can find matchings $\mathcal{M}'$ and $\mathcal{M}''$ of $\{2s+1, \dots, 6s\}$ and $\{6s+1, \dots, 10s\}$, respectively, that leave at most two elements of $\{2s+1, \dots, 10s\}$ unmatched.

Suppose first that all the elements of $\{2s+1, \dots, 10s\}$ are matched. By Lemma 2.2 with $r = s$, and $m = 3s$, the weight of $\mathcal{M}'$ is at least $(2s+1)+(2s+2)+\cdots+(4s-1)+4s$. Similarly, by applying Lemma 2.2 with $r = 3s$ and $m = 5s$, the weight of $\mathcal{M}''$ is at least $(6s+1)+(6s+2)+\cdots+(8s-1)+8s$. It is easy to see now that the sum of the weights of $\mathcal{M}'$ and $\mathcal{M}''$ is at least $n(n+1)/5$.

Observe now that if two elements of $\{2s+1, \dots, 10s\}$ are unmatched, then the sum of the weights of $\mathcal{M}'$ and $\mathcal{M}''$ decreases from $n(n+1)/5$ by at most $2n$. Hence the weight of $\mathcal{M}' \cup \mathcal{M}''$ is at least $n^2/5 - O(n)$. □

2.2 Point sets in general position

For point sets in general position, by discarding the elements with labels smaller than $n/3 + O(1)$ and applying Lemma 2.2, we obtain:

Theorem 2.5. *Let P be a labelled point set in general position. Then the heaviest non-crossing matching of P has weight at least $n^2/6 + \Omega(n)$.*

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