Sixth proof of the Orthogonal Art Gallery Theorem

Jorge Urrutia*†
(jorge@csi.uottawa.ca)

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Abstract

In this note we give a new and easy proof of the well known Art Gallery Theorem for orthogonal polygons, namely: any orthogonal polygon with n vertices can always be guarded with at most $\lfloor \frac{n}{4} \rfloor$ guards.

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1 Introduction

Art gallery, guarding or illumination problems have attracted much attention in recent years. The starting point here was the now well-known result that $\lfloor \frac{n}{3} \rfloor$ light sources are always sufficient to illuminate any polygon with n vertices; see [1, 2]. Our goal in this paper is to give a simple proof of the following well-known result, known as The Orthogonal Art Gallery Theorem proved originally by Kahn, Klawe and Kleitman [4]:

Theorem 1.1 $\lfloor \frac{n}{4} \rfloor$ *lights are always sufficient to illuminate any orthogonal polygon with n vertices.*

There are to our knowledge five different proofs of this result; see [3, 4, 5, 6, 7]. In this paper, we give yet another proof of Theorem 1.1. Our proof is simple and easy to understand.

A horizontal or vertical *cut* of P is the extension of a horizontal or vertical edge of P at a reflex vertex, towards the interior of P, until it hits the boundary of P. A horizontal or vertical cut is called an *odd cut* if it splits P into two non-empty polygons, P_1 and P_2, with n_1 and n_2 vertices respectively such that

[1]University of Ottawa, Department of Computer Science, Ottawa, ON, K1N 9B4, Canada.
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at least one of n_1 or n_2 equals $4k+2$ for some k. O'Rourke noted in his proof of the orthogonal art gallery theorem that if we can prove that any orthogonal polygon has an odd cut, then by an inductive argument, Theorem 1.1 would follow since if $n_1+n_2 \leq n+2$, we have $\lfloor \frac{n_1}{4} \rfloor + \lfloor \frac{n_2}{4} \rfloor \leq \lfloor \frac{n}{4} \rfloor$. The internal angles at the vertices of an orthogonal polygon are either of size $\frac{\pi}{2}$ or $\frac{3\pi}{3}$, and are called convex or reflex vertices respectively. A well-known fact that we will use here is that any orthogonal polygon with n vertices has exactly $\frac{n-4}{2}$ reflex vertices.

Given an orthogonal polygon P, we label its edges as *top*, *right*, *bottom*, and *left* in the natural way, and then call a reflex vertex of P a *top-left* reflex vertex if the edges of P incident to it are a top and a left edge. *Top-right*, *bottom-right*, and *bottom-left* edges are defined in a similar way; see Figure 1.

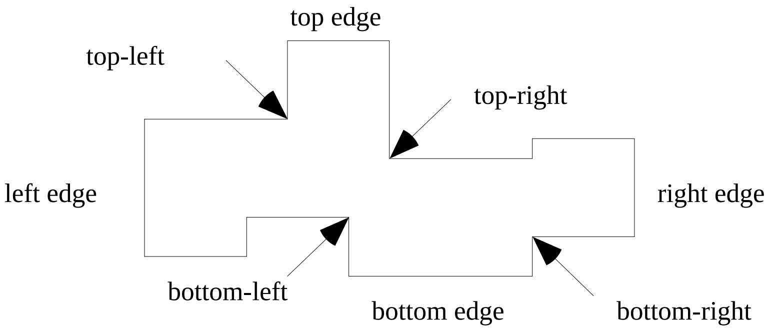

Figure 1: Classifying the edges and vertices of an orthogonal polygon.

We are ready to prove Theorem 1.1.

Proof: Our result is true for orthogonal polygons with 4 or 6 vertices. Let P be a polygon with at least 8 vertices. Split the set of reflex vertices of P into two sets; S_1 containing all the top-right and bottom-left reflex vertices of P, and S_2 containing all the top-left and bottom-right vertices. Since P has $\frac{n-4}{2}$ reflex vertices, one of S_1 or S_2 has at most $\lfloor \frac{n}{4} \rfloor$ vertices. Suppose it is S_1. If placing a light at every vertex of S_1 illuminates all of P, we are done. Suppose then that there is a point p in P not illuminated by S_1. Consider the longest horizontal line segment c containing p, and contained in P. Let e and f be the edges of P containing the endpoints of c; see Figure 2.

Consider the largest rectangle R containing c and contained in P. Let e' and f' be edges of P that intersect the top and bottom edges of R respectively. Since p is not visible from any point in S_1, it follows that e and e' meet at the top-left corner point of R. Similarly f and f' meet at the bottom-right corner point of R.

Let q and q' be the top-right and bottom-left vertices of R. If they are vertices of P, it follows that P is R and there is nothing to prove.

Otherwise, two cases arise:

1. Neither of of q and q' are vertices of P.
2. Exactly one of them, say q, is a vertex of P.

If neither of q and q' are vertices of P, it is easy to see that we can make two horizontal cuts that generate a rectangle contained in R and containing c as shown in Figure 2. It now follows that one of these two horizontal cuts is an odd cut of P.

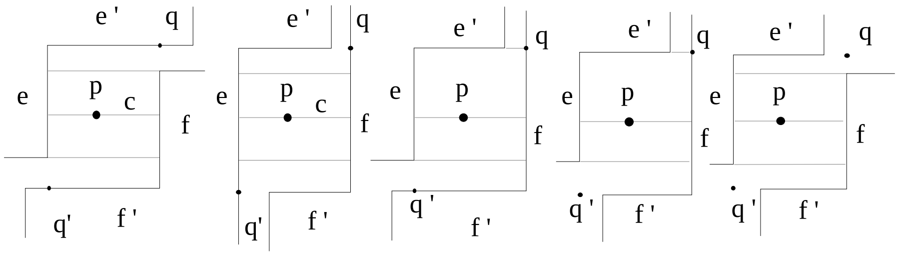

Figure 2: Up to symmetry, these are all the cases arising when none of q and q' are vertices of P.

Suppose then that only q is a vertex of P. If e and f' are properly contained in the left and bottom edge of R, then by extending the horizontal edge of P, incident to the bottom vertex of e, and the vertical edge of P incident to the left endpoint of f', we obtain a polygon with $n-4$ vertices that by induction can be guarded with $\lfloor \frac{n-4}{4} \rfloor$ guards. Since R can be guarded with a single guard, our result follows; see Figure 3.

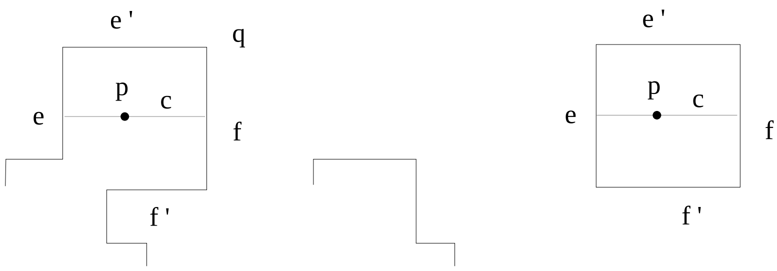

Figure 3: The case when e and f' are properly contained on the left and bottom edges of R.

Suppose then without loss of generality that f' properly contains the bottom edge of R and let w be the bottom vertex of e. Consider the horizontal line segment l joining w to a point in the base of R. Slide this segment until it hits a vertical edge of P or it reaches the left endpoint of f' or the leftmost endvertex of horizontal edge g incident to the bottom vertex of e. In the second case, if we reach the left endpoint of f', this point generates a vertical odd cut of P that leaves a polygon with six edges to its right, and another with $n-4$ on the left. In this case our result follows again by induction. The case when we reach the leftmost point of g follows in a similar way.

Suppose then that we hit a vertical edge of P. Let x be the highest vertex of P contained in l. Then we can generate two cuts of P at x, a horizontal cut h and a vertical cut h'. Let P' be the orthogonal subpolygon of P to the left of h' obtained when we cut P along h', and P'', the subpolygon on top of h obtained by cutting P along h. If P' has m vertices, P'' contains $m+2$ vertices, and thus either h or h' is an odd cut; see Figure 4. Our result follows. ■

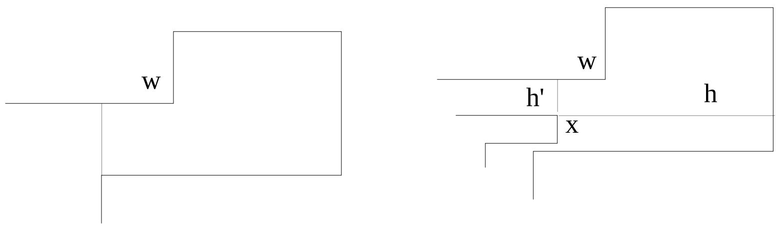

Figure 4: The case when f' properly contains the bottom edge of R.

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