K-Guarding Polygons on The Plane

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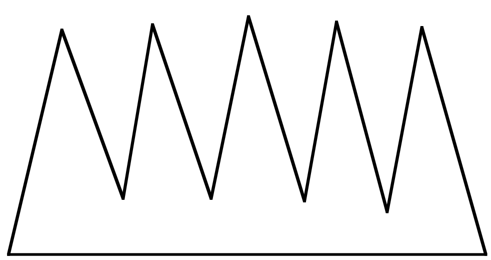
1. Introduction

Let P be a simple polygon with n vertices. We say that P is k-guardable if it is possible to find a set of points Q consisting of *interior points of edges of* P such that every point of P is visible from at least k elements in Q and *no edge of* P *has more than one element in* Q. The following question was asked by A. Lubiw at the open problem session of the Fourth Canadian Conference in Computational Geometry: For what values of k, is every simple polygon k-guardable? It has been observed by T. Shermer that *comb* polygons [Chv75, O'R87] are not 3-guardable; such a polygon is shown in Figure 1.



A polygon which is not 3-guardable

Figure 1.

In this paper we prove that every simple polygon with n vertices can be 2-guarded using at most n-1 points. We also prove that any simple polygon with n vertices can be 1-guarded with at most $\lfloor \frac{n}{2} \rfloor$ guards. These bounds are tight up to an additive constant. We prove that any polygon with one hole is also 2-guardable. We also prove that every polygon with holes is 1-guardable, and that it is not true that every polygon with holes is 2-guardable.

2. One and Two-Guarding Simple Polygons.

In this section, we consider the problem of 1-guarding and 2-guarding simple polygons. We proceed now to prove our first result, namely that every simple polygon can be 2-guarded.

Theorem 1: Every simple polygon can be two-guarded.

Proof: Let a be any point on the interior of an edge of P and let P_a be the *visibility polygon of* a, that is the set of all points $q \in P$ such that the line joining a with q is contained in P*. Notice that P_a may contain vertices that are not vertices of P and that some edges of P may have up to two vertices of P_a in their interior (See Figure 2.) Let v be a vertex of P_a that is not a vertex of P. The line joining v to a contains a vertex of P, which we shall denote by v_a. Let e be an edge of P that has two vertices of P_a in its interior, say b and c. Notice that b and c_a are mutually visible in P_a (the triangle formed by a, b and c is contained in P_a.) Thus the line segment joining

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* To facilitate our presentation, we may assume that a is not contained in the line joining any two vertices of P and that for every edge e of P the line containing e contains no vertex of P other that the endvertices of e. This condition may be easily dropped, leaving our result unchanged.

them is contained in P_a. Remove from P_a the triangle determined by b, c and c_a. Apply this procedure to all edges of P containing two vertices of P_a and call the resulting polygon P_a^1. Place a guard at all vertices of P_a^1 that are not vertices of P_a. If an edge e of P is completely visible from a place one guard in the middle of it and finally place one guard at the point a itself (See figure 2.).

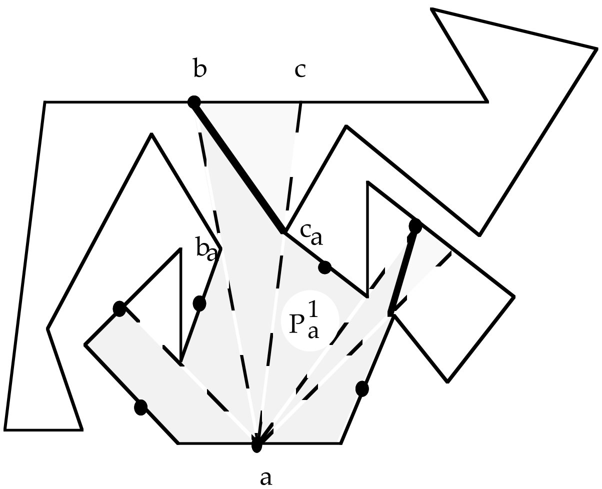

Figure 2

Observation 1: All points in P_a^1 are 2-guarded (by a and at least one of the other guards placed on the boundary of P_a^1.

Clearly $P-P_a^1$ can be "broken" into several simple polygons $P_1,\ldots,P_k$ with disjoint interiors with the property that each one of them contains exactly one vertex that is not a vertex of P. We will denote such a vertex by v(i), i=1,...,k. Notice that some pairs of elements of $P_1,\ldots,P_k$ may have at most one point in common, i.e. a vertex of P_a^1 that is not a vertex of P. Now we process recursively each P_i using the following recursive procedure:

Procedure 2-Guarding (P_i, v(i))

Calculate its visibility polygon $P_{v(i)}$ of v(i) in P_i. Two cases arise:

a) $P_i=P_{v(i)}$. In this case place a guard in the middle of each edge of $P_{v(i)}$ except for all the two edges of $P_{v(i)}$ containing v(i).

b) $P \neq P_i$. Three cases are consider now:

i) An edge e of P_i completely visible from v(i). Place a guard in the middle of e.

ii) An edge of P_i containing exactly one vertex v of $P_{v(i)}$ that is not a vertex of P_i. Place a guard at v.

iii) For each edge e of P_i containing two vertices of $P_{v(i)}$ say b and c that are not vertices of P_i proceed as follows: Locate the reflex vertices $b_{a(i)}$ and $c_{v(i)}$ of P_i contained in the interior of the line segment joining v(i) to b and c respectively. Join $b_{a(i)}$ to c with a line segment and delete from $P_{v(i)}$ the triangle with vertices b, c and $b_{a(i)}$. Place a guard at c. Let $P_{v(i)}^1$ be the polygon obtained from P_i after we deleted all the triangles generated by edges containing two vertices of $P_{v(i)}$ not vertices of $P_{v(i)}$. Partition $P_{v(i)}-P_{v(i)}^1$ into m simple polygons $P_1,\ldots,P_m$ each containing exactly one vertex v(j) that is not a vertex of P_i, j=1,...,m.

For j=1,..., m execute **2-Guarding** (P_j, v(j)).

End 2-Guarding

It now follows by Observation 1 that the collection of guards thus obtained is a 2-guarding of P, that is each visibility subpolygon $P_{v(i)}$ calculated during our execution of **2-Guarding** is 2-guarded. Moreover, our procedure places at most one guard on each edge of P.

QED.

Corollary 1: $\lfloor \frac{n}{2} \rfloor$ guards are always sufficient and sometimes necessary in a 1-guarding of a simple polygon.

Proof: In the proof of Theorem 1, color the initial point a with color 1 and the guards generated by a with color 2. In the successive iterations, if a guard was generated by a guard with color 1 (resp. 2), color it with color 2 (resp. 1). By observation 1, and our coloring rule, it follows that every point is seen by at least one point with color 1 and one with color 2. Choose the color class with fewer vertices to obtain the sufficiency of our result. The family of comb polygons similar to the polygon shown in Figure 1 demonstrates that $\lfloor \frac{n}{2} \rfloor$ guards are sometimes required.

QED.

3. Polygons With Holes

Given a simple polygon P', and k disjoint polygons $Q_1,\ldots,Q_k$ contained in the interior of P, we say that the polygon $P=P'-(Q_1 \cup \ldots \cup Q_k)$ is a polygon with k holes. An edge e of P' will be called an exterior edge of P while edges of $Q_1,\ldots,Q_k$ will be called internal edges of P.

In this section we study the problem of 1 and 2-guarding for polygons with holes. We start by proving:

Theorem 2: Not every polygon with holes is 2-guardable.

Proof: To prove Theorem 2, all we have to do is to exibit a polygon with two holes that is not 2-guardable. To this end consider the polygon with two holes shown in Figure 4.

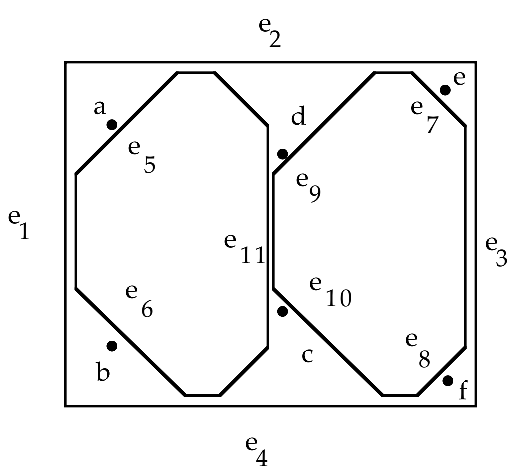

Figure 4

Consider the point set S={a,b,c,d,e,f}. In order to two-guard the elements of S, we can choose guards only placed in the interior of $e_1,\ldots,e_{11}$. Moreover, no guard placed in any of these edges can see two elements of simultaneously. Our result now follows.

QED.

Next we prove:

Theorem 3: Every polygon with holes is 1-guardable.

Before we proceed with the proof of Theorem 3 we recall the following result on visibility.

Lemma 2: Let $S=\{\sigma_1,\ldots,\sigma_n\}$ be a collection of disjoint line segments and p a point on the plane such that p is externally visible from S, i.e. there is a ray starting at p that does not intersect any one of the elements of S. Then S contains at least one line segment σ_i that is completely visible from p.

A proof of this Lemma can be obtained from results presented in [FRU]. It is easy to see that p induces an order relation "<" in S as follows:

i) We say that σ_a blocks σ_b (denoted by $\sigma_a \to \sigma_b$) if there is a point q in σ_b such that the line segment joining p to q intersects σ_a.

ii) We now say that $\sigma_a < \sigma_b$ if $\sigma_a \to \sigma_b$ or there is a chain of elements $\sigma_a = \sigma_1 \to \sigma_2 \ldots \to \sigma_k = \sigma_b$.

In the language of [FRU] "<" is a light source order. Thus the element σ_i claimed in Lemma 2 is nothing else than a minimal element of the order relation "<" on S.

Proof of Theorem 3: Let P be a polygon with holes. Without loss of generality, assume that no edge of P is parallel to the x-axes, that no two vertices of P have the same y-coordinate, and that the difference between the y-coordinate of any two such vertices is at least $\varepsilon > 0$.

For every vertex v of P consider the longest line segment contained in P which is parallel to the x-axis and contains v. These lines partition P into a collection of convex polygons $T=\{R_1,\ldots,R_m\}$ with disjoint interiors. For every edge e of P place a guard in its interior at distance at most $\frac{\square}{2}$ from its lower end point.

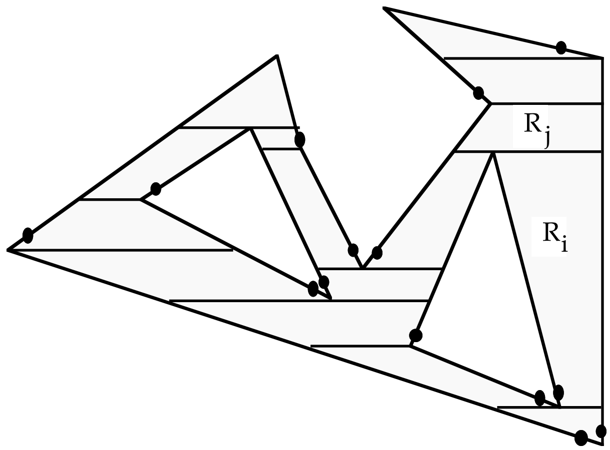

Figure 5

We claim that these points 1-guard P. In order to prove our claim we observe that if the boundary of a region R_i of T intersects the interior of an edge e and also contains its lower end-point, then it contains the guard assigned to e. Suppose then that an element R_j of T does not contain a guard in its boundary and consider a point p in R_j. If p lies in a line segment contained in P that contains an edge e of P, then the guard assigned to e guards p. Suppose then that this is not the case. Using the horizontal line through p cut the polygon P in two parts and delete that part of P above it. At all the remaining vertices of P, cut away a sufficiently small segment from each edge of P, or the remaining segment of an edge of P (See Figure 6). Notice that we get a disjoint family of line segments for which p is externally visible. By Lemma 2, one of these segments say e' is completely visible from p. Since p is in the interior of P, it follows that p sees the side of e' facing towards the interior of P, and thus the guard assigned to the edge of P that contains e' guards p.

To finish this paper we give an outline of our last result in this paper, namely:

Theorem 4: Every polygon with exactly one hole is 2-guardable

An edge e of a simple polygon P is called *convex* if the end vertices of e are convex vertices of P. We give the following lemmas without proof.

Lemma 3: Let e be a convex edge of P. Then there is a 2-guarding of P with n-1 guards that does not use a guard at e. Furthermore, such guarding can be chosen so that the guards at the edges adjacent to e are arbitrarily close to e.

Proof: Let us assume that e is convex, and that the edges adjacent to e in P are e_{i-1} and e_{i+1}. In the proof of Theorem 1, choose the initial point a on ei-1 and close enough to ei such that a sees an interior point of e_{i+1}. See Figure 5. In this case, it is easy tu see that the point x that our initial procedure in theorem 1would place on e_i is redundant, (See Figure 5). If we now proceed as in the rest of theorem 1, we end up with a two guarding of P that does not place any guard at e_i.

To prove our second claim of this lemma, we notice first that a can be chosen arbitrarily close to e. This however, could places the guard at e_{i+1} at a point p at fixed distance $\square$ from e as in Figure 5. In this case, e_{i+1} is not entirely visible from a. Then the line joining a to the guard y in e_{i+1} contains a vertex, say v in its interior. To solve this case, we simply choose a point w on e_{i+1} as close to e as we choose, place aguard at w and delete from P^1_a the triangle with vertices v, w and p. From here on we proceed as in Theorem 1 (See figure 6).

QED.

A *funnel* is a polygon P with vertices $v_1,\ldots,v_n$ such that P contains exaclty three convex vertices, v_1, v_k and v_n for some 1<k<n.

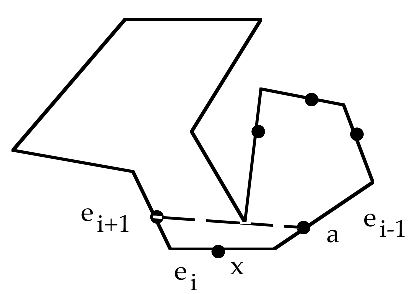

Figure 5

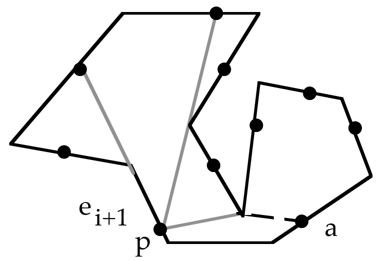

Figure 6

Let $e = [v_i, v_{i+1}]$ an edge of P and $v_k \neq v_i, v_{i+1}$ be a vertex of P . Then it is easy to see that if the shortest polygonals paths H_i and H_{i+1} contained in P from v_k to v_i and v_{i+1} are disjoint, then the polygon bounded by e, H_i and H_{i+1} is a funnel; denote this funnel by F(e, v_k). We now have:

Lemma 4: Let $e = [v_n, v_1]$ be a convex edge of a polygon P and $v_k \neq v_1, v_n$ a vertex of P such that the following conditions are satisfyed:

a) The shortest polygonals in P from v_k to v_n and v_1 are disjoint.

b) P-F(e,v_k) can be broken into s interior disjoint subpolygons $P_1,..,P_s$ of P such that the edge e_j of P_j that intersects F is a convex edge in P_j; i=1,...,s.

Then P can be two guarded avoiding placing a guard at e.

Proof(Sketch): Place a guard in the middle of each edge of F(e, v_k) that is an edge of P. By Lemma 3, every one of $P_1,...,P_s$ can be two guarded avioding placing guards at the edges of F(e, v_k). It is now easy to see that if in each P_i the guards placed at the edges incident with e_i are placed close enough to e_i then these guards together with those placed at the edges of F(e v_k) that are edges of P will also 2-guard F(e, v_k).

QED.

The next observation will be useful:

Observation 1: The guards placed at the edges adjacent to v_k can be placed arbitrarily close to v_k. Moreover, the guards placed at the edges adjecent to edege e can also be placed arbitrarily close to e.

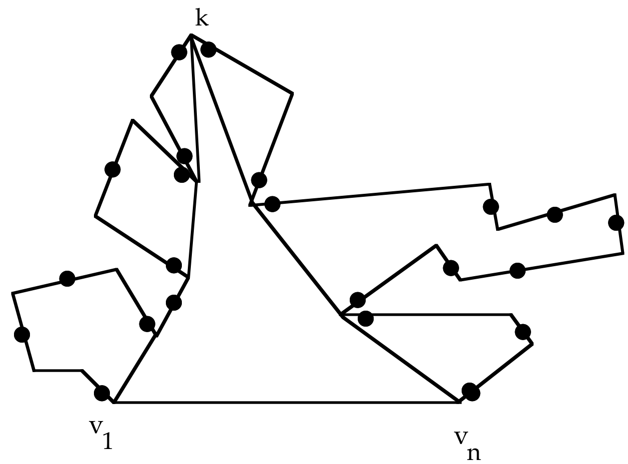

Figure 6

We are now ready to prove Theorem 4.

Proof of Theorem 4: Let P be a polygon with one hole. Consider two points u and v that are mutually visible in P such that u is an interior point to an external edge of P while v is an interior point to an internal edge of P.

Delete from P an □-stripe L alog the line segment joining u to v to obtain a new simple polygon P" with four new vertices and two new *convex edges* e' and e". We now proceed to two guard P" as in Theorem 1 but modify our procedure to avoid placing guards on e' and e".

Since e' ia a convex edge of P", by lemma 3, we can avoid placing a guard at e' by selecting the initial guard on any of the two edges of P" adjacent to e'. Moreover, the second guard placed at the other edge adjacent to e' can also be chosen arbitrarily close to e'. We proceed now to 2-guard P" until we generate the first guard call it x that can see a point of e". At this point, calculate the funnel F(e",x) generated by e" and x in P". Let y be the guard that generayed x. By cutting P" along the line joining y to x, we obtain two polygons one of which, call it P"(e") contains edge e" of P". Finish the 2-guarding procerdure for P"-P"(e") first. We now proceed to 2-guard P"(e"). By Lemma 4, we can 2-guard P"(e") in such a way that

a) No guard is placed at e" and the guards placed at the edges of P" adjacent to e" are arbitrarily close to e".

b) The guards placed at the edges of P"(e") adjacent to x are arbitrarily close to x. At this point, eliminate these two guards and substitute them for a gurd at x. Notice that we still get a two guarding of P"(e").

We now obtain a two guarding of P' by simply using the 2-guarding of P'-P(e") and the 2-guarding of P"(e") (See Figure 7).

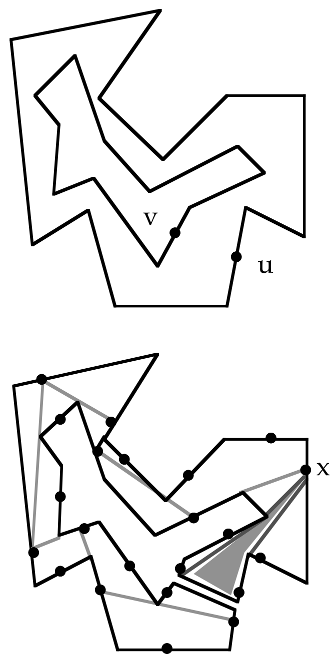

Figure 7

To finish our result, we notice that the guards on the edges of P' adjacent to e' and e" can be placed arbitrarily close to them, and thus we can replace them by guards at he original points u and v of P to obtain a two guarding of P.

QED.

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