A Containment Result on Points and Circles

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Abstract

Let P_n be a collection of n points on the plane. For any x,y,z □ P_n let C(x,y,z) be the number of elements of P_n contained in the circle through x, y and z. Let $A(P_n)$ be the average value of C(x,y,z) over all triples of points {x,y,z} contained in P_n. In this paper we prove that for any collection of points P_n, $A(P_n) \geq$ □(n-3)/3.33...□, that is the *expected* number of elements of P_n contained in any circle through three points in P_n is at least □(n-3)/3.33...□. For the case when the elements of P_n are the vertices of a convex polygon, $A(P_n) =$ □(n-3)/2□. In this case our bound is tight.

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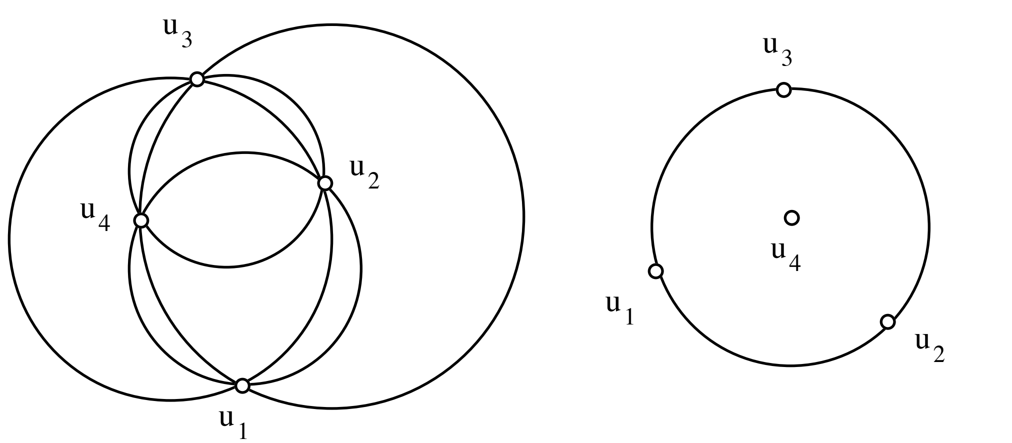
1. Introduction

Let P_n be a collection of n points on the plane, no three of which are aligned, nor any four of which are cocircular. The following result was proved in [NU]: For any P_n there are two points $x,y \in P_n$ such that any circle containing x and y contains at least $\lceil (n-2)/60 \rceil$ points in P_n. This result has been subsequently improved in a sequence of papers; to $\lceil n/27 \rceil+2$ in [HRW], to $\lceil n/30 \rceil$ in [BSSU], to $\lceil 5(n-2)/84 \rceil$ in [H] and most recently to $\lceil n/4.7 \rceil$ in [EHSS]. In this paper, we prove the following related result. For any P_n on the plane and x, y, z $\in P_n$ the expected number of points of P_n contained in the circle through x, y and z is at least $\lceil (n-3)/3.33.. \rceil$. For the convex case, i.e. when the points in P_n are vertices of a convex polygon, our result can be improved to $\lceil (n-3)/2 \rceil$. For this case, this bound is optimal.

2. Preliminary Results

Consider any collection P_n of n points on the plane. Recall that for the rest of this paper we will assume that no three elements of P_n are aligned, nor four of them are cocircular. For any subset {x,y,z} of P_n let C(x,y,z) be the number of points of P_n contained in the circle through x,y and z (see Figure 1).



(a) $C(u_1, u_2, u_3)=1, C(u_1, u_2, u_4)=0$

(b) $C(u_1, u_2, u_3)=1, C(u_2, u_3, u_4)=0$

Figure 1

Denote by $A(P_n)$ the average value of C(x,y,z) taken over all {x,y,z} contained in P_n. The main objective in this section is to prove the following result:

Theorem 1: $A(P_n) \geq$ □$(n-3)/3.33..$□

Some preliminary results will be needed to prove Theorem 1.

Consider a subset of P_n with exactly four elements $\{u_1, u_2, u_3, u_4\}$. Two possibilities arise for the convex closure conv$\{u_1, u_2, u_3, u_4\}$ of $\{u_1, u_2, u_3, u_4\}$; either conv$\{u_1, u_2, u_3, u_4\}$ is a triangle or conv$\{u_1, u_2, u_3, u_4\}$ is a quadrilateral. Consider the four circles C_i determined by $\{u_1, u_2, u_3, u_4\} - \{u_i\}$, $i=1,...,4$.

Observation 1: If conv $\{u_1, u_2, u_3, u_4\}$ is a triangle, *exactly one* of the four circles C_i, $i=1,...,4$, will contain the four points in $\{u_1, u_2, u_3, u_4\}$ (see Figure 1 (b)). If conv$\{u_1, u_2, u_3, u_4\}$ is a quadrilateral, then *exactly two* of the four triangles C_i, $i=1,...,4$, will contain the four points in $\{u_1, u_2, u_3, u_4\}$ (see Figure 1 (a)).

In the second case, if u_2 and u_4 are opposite to each other and the sum of the angles at u_2 and u_4 is at least 180° then the circle through u_1, u_2 and u_3 contains u_4 in its interior and the triangle through u_1, u_3 and u_4 contains u_2 in its interior (see Figure 1 (b)).

Construct a bipartite graph $B(P_n)$ whose vertex set consists of the three and four subsets of P_n. A three subset S of P_n is adjacent to a four subset S' of P_n iff S is contained in S' and the circle through the points in S contains the four points in S' (see Figure 2).

Observation 2: In $B(P_n)$ any vertex representing a four set S' has degree one or two, depending on whether conv(S') is a triangle or a quadrilateral.

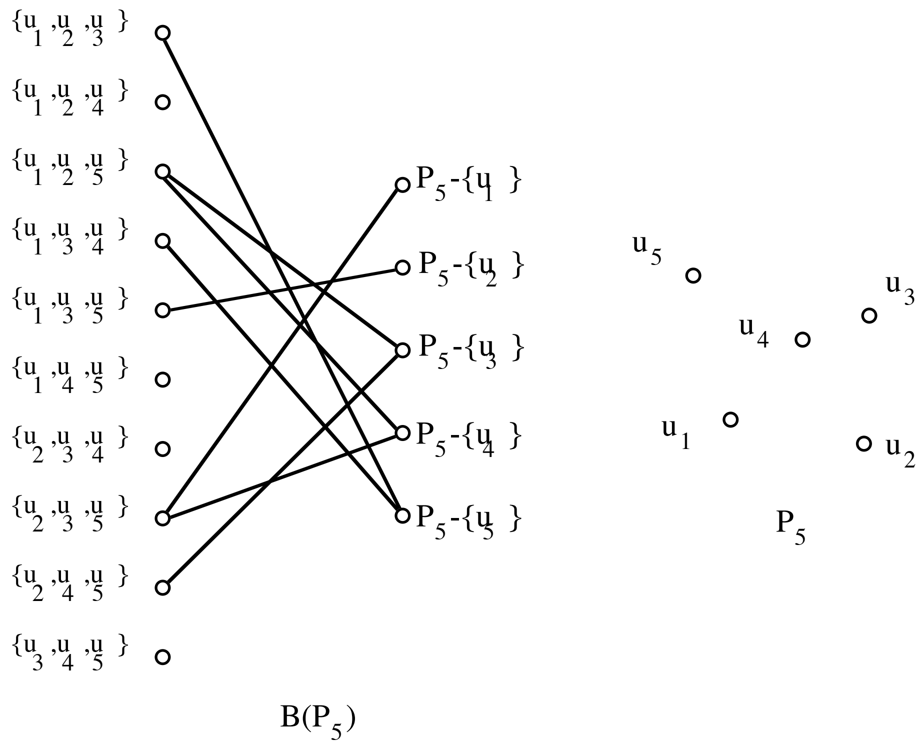


Figure 2

Lemma 1: If the degree deg(S) of a three set S={x,y,z} in $B(P_n)$ is k, then C(x,y,z)=k, i.e. the number of points in P_n contained in the interior of the circle through the points in S is exactly k .

Proof: Let S be a three subset of P_n and H(S) the circle through the points in S. For every point x in the interior of H(S), the sets S and $S'=S\cup\{x\}$ are adjacent in $G(P_n)$. The result now follows easily.

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Let a_1 be the number of four subsets S' of P_n whose convex closure conv(S') is a triangle and a_2 the number of four subsets S' of P_n for which conv(S') is a cuadrilateral.

Lemma 2: The sum of the degrees of all the vertices of $B(P_n)$ representing three subsets S of P_n equals $a_1+2a_2=\binom{n}{4}+a_2$.

Proof: By Observation 2, each four subset S' of P_n contributes one or two to the sum of the degrees of the vertices of $B(P_n)$ representing three subsets of P_n, depending on whether conv(S') is a triangle or a quadrilateral.

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For example in Figure 2, conv(P_5-{u_1}) and conv(P_5-{u_2}) are both triangles and conv(P_5-{u_3}), conv(P_5-{u_4}) and conv(P_5-{u_5}) are 4-gons. Then for this case a_2=2 and a_2=3. Then the number of edges in $G(P_5)= a_1+2a_2=2+3*2=\binom{5}{4}+a_2=8$. Also the degree in $B(P_5)$ of each vertex representing a three S subset of P_5 is the number of points contained in the circle through S.

It now follows that the average value $A(P_n)$ we are trying to determine is the average degree in of the vertices $B(P_n)$ representing three subsets of P_n. Then we can obtain the following equality for $A(P_n)$:

Corollary 1: $A(P_n)=\dfrac{\binom{n}{4}+a_2}{\binom{n}{3}}$.

3. The Convex Case

We now proceed to prove our first result for the case when the elements of P_n are the vertices of a convex polygon. A set of points P_n will be called convex if the elements of P_n are the vertices of a convex polygon.

Theorem 2: If P_n is a convex set of points, $A(P_n)=(n-3)/2$

Proof: If P_n is a convex set, for any 4-subset S' of P_n conv(S') is a cuadrilateral. Then $a_2=\binom{n}{4}$ and by Lemma 2 and Corollary 1 $A(P_n)=\dfrac{2\binom{n}{4}}{\binom{n}{3}}=(n-3)/2$.

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4. The General Case

We now proceed to obtain a lower bound for the value of $A(P_n)$ for the general case when P_n is not necessarily convex. Notice that Theorem 2 gives us an upper bound for the value of $A(P_n)$ for the general case when P_n is no necessarily convex.

To obtain a lower bound for $A(P_n)$, by Corollary 1, all we need to do is to establish a lower bound for the value of a_2, i.e. a lower bound on the number of four subsets of P_n whose convex closure is a quadrilateral.

To do this let us construct a graph $G(P_n)$ from P_n as follows. The vertices of $G(P_n)$ are all the two subsets of P_n. Two subsets {u,v} , {x,y} are adjacent in $B(P_n)$ iff the line segments l(x,y) and l(u,v) joining x to y and u to v intersect (see Figure 3). Let $I(P_n)$ be the number of edges in $G(P_n)$.

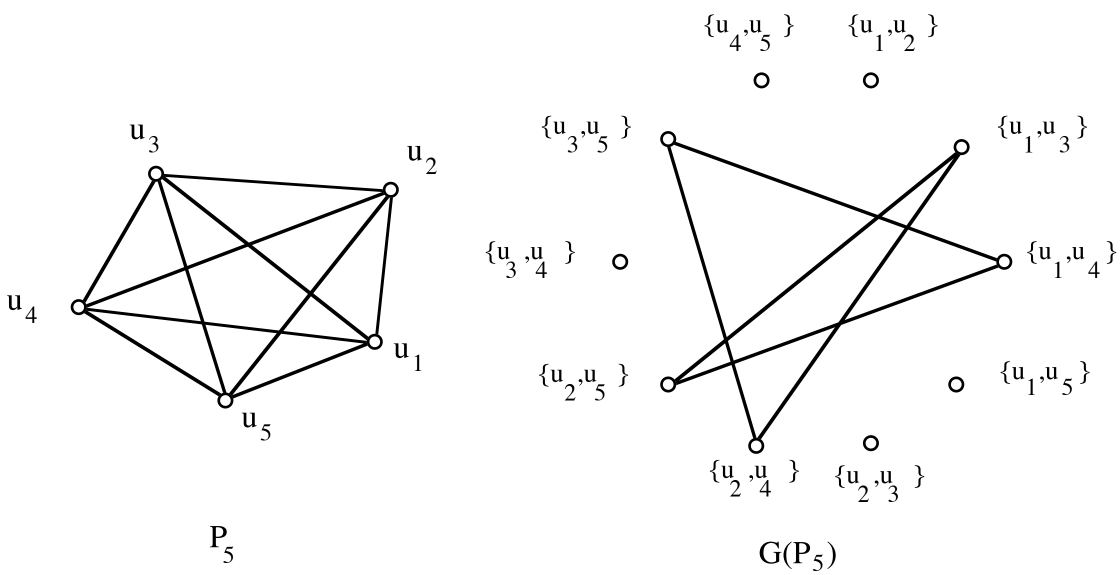

Figure 3

Lemma 3: $a_2=I(P_n)$.

Proof: If {u,v} , {x,y} are adjacent in $I(P_n)$, then (x,y) and l(u,v) intersect and conv(u,v,x,y) is a quadrilateral. Conversely each subset S' of P_n whose convex closure is a quadrilateral determines exactly one edge in $I(P_n)$.

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The following result was proved in [NU].

Lemma 4: $I(P_n)$ is at least $\frac{\binom{n}{5}}{n-4}$.

4. The Main Result

We are now ready to prove our main result.

Proof of Theorem 1

By Corollary 1, $A(P_n)=\frac{\left(\binom{n}{4}+a_2\right)}{\binom{n}{3}}$. Using Lemmas 3 and 4, we obtain that

$a_2=I(P_n)\geq \frac{\binom{n}{4}}{(n-4)}$. Then $A(P_n)\geq \frac{\left(\binom{n}{4}+\left(\frac{\binom{n}{5}}{(n-4)}\right)\right)}{\binom{n}{3}}$ =6(n-3)/20 = (n-3)/3.33... .

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As a consequence of our results we can easily prove the following result:

Theorem 3: Let P_n be any collection of points on the plane, $u,v \in P_n$ and C any circle through u and v containing at least a third point x in P_n. Then the expected number of points of P_n contained in C is at least $\lceil (n-3)/3.33.. \rceil$. If P_n is convex the above bound can be improved to $\lceil n/2 \rceil$.

Proof: Let C ba a circle through u and v containing a subset H of P_n. Then it is easy to verify that one of the following two possibilities holds:

a) there is a third point $w \in P_n$-H such that the circle C' throuh u, v and w contains in its interior the same set of points in P_n as C.

b) there is a third point $w \in$ H such that the circle through u,v and w contains in its interior exacly the elements in H-w.

Our result now follows in a similar way as Theorem 1.

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5 Conclusions

As we pointed out in the introduction of this paper, it is known that for any collection of points on the plane there are $u,v \in P_n$ such that any circle through u and v contains at least $\lceil n/4.7 \rceil$ points in P_n. On the other hand, the results presented in this paper tell us that for any three points $u,v,w \in P_n$ the expected number of points of P_n contained in the circle through them is at least $\lceil (n-3)/3.33.. \rceil$. We also show that for any two points u,v P_n the expected number of points contained in any circle through u and v (containing at least a third point x in P_n) is at least $\lceil (n-3)/3.33.. \rceil$ For the convex case, the both bounds are improved to $\lceil n/2 \rceil$.

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