A Combinatorial Result on Points and Circles on the Plane

V. Neumann-Lara*
and
J. Urrutia**

Abstract

Let P_n be a collection of n points on the plane. For a pair of points u and v P_n let C(u,v) be the minimum number of points of P_n contained in any circle contaning u and v. In this paper we prove the result that there exist two points u_o and v_o P_n such that any circle containing u_o and v_o contains at least $\lceil(n-2)/60\rceil$ elements of P_n (other than u_o and v_o). We also prove that the average value of C(u,v) over all pairs {u, v} P_n is ≥ $\lceil(n-2)/60\rceil$. For the case when P_n are the vertices of a convex polygon, we prove that there exist two vertices u, v of P_n such that any circle containing them contains at least $\lceil(n-2)/4\rceil$ elements of P_n.

* Universidad Nacional Autonoma de Mexico, Instituto de Matemáticas.
**University of Ottawa, Department of Computer Science.

Introduction

Geometry is one of the most interesting and important branches of mathemat- ics. The origins of geometry can be traced back to many ancient cultures, such as that of the Greeks. Some of the most important studies in geometry, such as the famous *Elements of Euclid,* deal simply with properties of collections of points, lines and circles on the plane. With the advent of combinatorics, new types of results in geometry are being obtained, results which involve the <u>combinatorial</u> properties of sets of points, lines and circles on the plane. See for example, [2], [5], [6], [11]. The interested reader can find an excellent source of results of this nature in Hadwiger, Debrunner and Klee's book *Combinatorial Geometry on the Plane* [10] and Grunbaum's *Arrangements and Spreads* [7].

In this paper, we prove the following result: Given a collection P_n of n points on the plane, there exist two points u, v of P_n such that any circle containing them contains at least $\lceil n-2/60 \rceil$ points in P_n. Furthermore let C(u,v) be the minimum number of points of P_n contained in any circle containing u and v; u,v $\in P_n$. Then we prove that the average value of C(u,v) over all pairs $\{u,v\} \subseteq P_n$ is at least $\lceil n-2/60 \rceil$. The proof of this result uses basic results in combinatorics and graph theory.

For the sake of clarity, all collections P_n of points considered in the rest of this paper do not contain any three aligned points. This restriction can be easily lifted, leaving the results of this paper unchanged.

Preliminaries

A graph G = (V(G), E(G)) consists of a collection of points V(G) called the vertices of G and a set E(G) of unordered pairs of elements of V(G) called the edges of G. If the pair {u, v} belongs to E(G), we say that u and v are adjacent. An edge {u, v} will be denoted by u—v. We say that u—v joins u to v. In this paper, we will deal with representations of graphs on the plane. In these representations, the vertices of G are represented by points on the plane, and the edges u—v of G by open segments of lines joining the points u and v. Let Bin(r,s) denote the binomial coefficient of r and s.

A complete graph K_n is a graph with n vertices such that for any pair of vertices u, v of $V(K_n)$, u—v is an edge of K_n. The following well-known result of graph theory will be used:

Theorem 0: Any planar representation of K_5 contains two edges which intersect at a point p not a vertex of K_5.

For a pair of points u and v on the plane let l(u,v) be the open segment of line joining u to v. The following lemma will prove useful in the proof of our main result.

Lemma 1: Let u, v, x, y be points in the plane such that l(u,v) and l(x,y) intersect. Then any circle containing u and v contains at least one end point of l(x,y) or any circle containing x and y contains at least one end point of l(u,v).

Proof: It is easy to see that u, v, x and y form the vertices of a convex quadrilateral. (See Figure 1.)

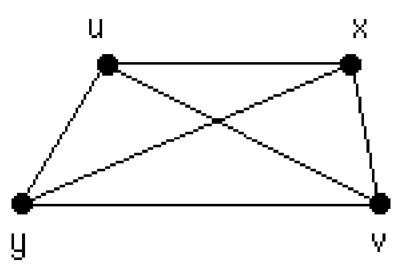

Figure 1.

In any such quadrilateral there exist two internal opposite angles such that their sum is greater than or equal to 180°. Without loss of generality let us assume that it is the angles of u and v that satisfy this property. Then any circle containing x and y contains at least one of u or v.

Let P_n be a collection of n points on the plane. An imbedding of K_n on the plane can be obtained by representing the vertices of K_n with the points of P_n and the edges of K_n with the segments l(u,v), u≠v, u, v $\in P_n$.

Let us define $I(P_n)$, the intersection number of P_n as follows: $I(P_n)$ is the number of different segments l(u,v), l(x,y) such that l(u,v) $\cap$ l(x,y)≠Ø; u, v, x, y $\in P_n$.

Lemma 2: $I(P_n) \geq$ Bin(n,5)/(n-4).

Proof: By Theorem 0, for each subset S of P_n with exactly five elements, there exist four points u, v, x, y $\in$ S such that $l(u,v) \cap l(x,y) \neq \emptyset$. Furthermore, the subset {u, v, x, y} appears in exactly n-4 subsets of P_n with five elements.

Finally, let us define a graph $G(P_n)$ (the intersection graph of P_n) as follows: $V(G(P_n)) = \{l(u,v);\ u, v \in P_n,\ u \neq v\}$ and two vertices l(u,v), l(x,y) of $G(P_n)$ are adjacent if $l(u,v) \cap l(x,y) \neq \emptyset$. See Figure 2.

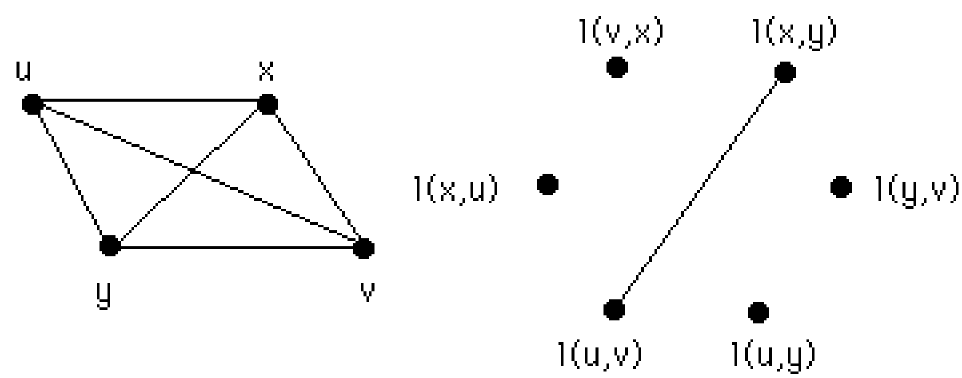

Figure 2.

Clearly $G(P_n)$ has exactly $I(P_n)$ edges, one for each pair of intersecting segments l(u,v), l(x,y).

Corollary 1: $|E(G(P_n))| \geq Bin(n,5)/(n-4)$.

We now obtain an orientation $G^*(P_n)$ of $G(P_n)$ as follows: An edge l(u,v)—l(x,y) is oriented l(u,v)—>l(x,y) if any circle containing u and v contains x or y, otherwise we orient l(x,y)—>l(u,v). This orientation is consistent because of Lemma 1. We should notice that if u, v, x and y lie on a circle, then we could choose either orientation; l(u,v)—>l(x,y) or l(x,y)—>l(u,v).

Let $d^+(u)$ be the out-degree of a vertex u in a directed graph D, ie. the number of vertices x such that u—>x $\in$ E(D).

Lemma 3: There exists a vertex $l(u_o,v_o)$ of $V(G^*(P_n))$ such that

$d^+(l(u_o,v_o)) \geq \lceil(n-2)(n-3)/60\rceil$.

Proof: From Corollary 1, $|E(G^*(P_n))| \geq Bin(n,5)/(n-4)$.

We also know that

$$\sum d^+(l(u,v)) = |E(G^*(P_n))| .$$

Then

$$\sum d^+(l(u,v)) \geq Bin(n,5)/(n-4).$$

Since we have exactly Bin(n,2) vertices in $G^*(P_n)$, there exists a vertex $l(u_o,v_o)$ with

$$d^+ (l(u_o,v_o)) \geq \left[Bin(n,5)/(n-4)\right] / Bin(n,2) = (n-2)(n-3)/60.$$

Results

We are now ready to prove our main result.

Theorem 1: For any collection P_n of n points on the plane, there exists a pair of points u_o, v_o such that __any__ circle containing them contains at least $\left|{}^{n-2}/_{60}\right|$ points of P_n.

Proof: By Lemma 3, there exist two points, u_o, $v_o \in P_n$ such that $d^+(l(u_o,v_o)) \geq (n-2)(n-3)/60$. This means that there exist at least $\lceil(n-2)(n-3)/60\rceil$ segments $l(x,y)$, $x, y \in P_n$ such that any circle containing u_o and v_o contains one end point of each one of these $(n-2)(n-3)/60$ segments. Eliminating redundancies (each point could appear in at most n-3 such segments), we have that any circle containing u_o and v_o contains at least $\lceil(n-2)/60\rceil$ points of P_n different from u_o and v_o .

In Lemma 3, we showed that there exists a vertex $l(u,v)$ in $G(P_n)$ with $d^+(l(u,v)) \geq (n-2)(n-3)/60$. Using the same arguments as for Lemma 3, we can obtain the following corollary.

Corollary 2: The average out-degree $d^+(l(u,v))$, $l(u,v) \in V(G(P_n))$ is greater than or equal to $(n-2)(n-3)/60$.

For a pair of points $u, v \in P_n$, let $C(u,v)$ be the minimum number of points of

P_n that are contained in any circle containing u and v. Let $A(P_n)$ be the average over all C(u,v), $u, v \in P_n$. A result which is much stronger than Theorem 1 now follows:

Theorem 2: $A(P_n) \geq (n-2)/60$.

For the special case when P_n are the vertices of a convex polygon, we can prove the following result:

Theorem 3: Let P be a convex polygon. Then there exist two vertices u and v of P such that any circle containing them contains at least $\lceil (n-2)/4 \rceil$ vertices of P.

Proof: We prove the theorem for the case when n=2k; the case n=2k+1 can be proved in a similar way. Let us label the vertices of P by 0,1,...,n-1 by tracing the boundary of P in the clockwise direction (see Figure 3).

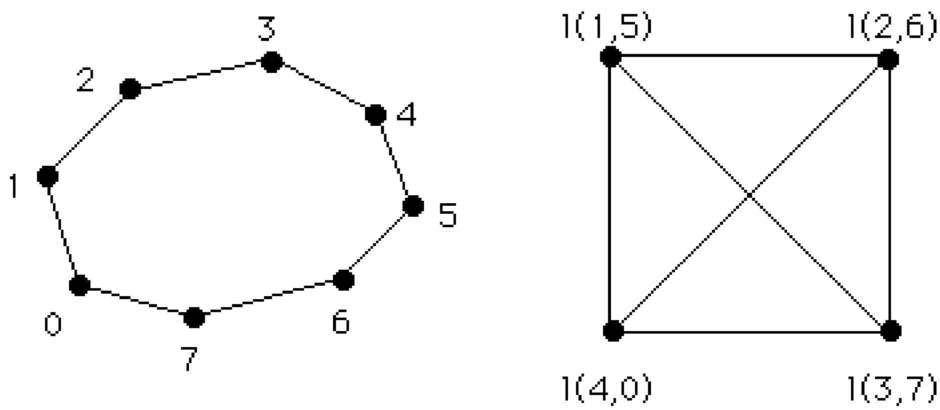

Figure 3.

Let G be the intersection graph of the segments of lines joining opposite pairs of vertices of P, ie. V(G)={l(i,i+1); i+k mod n}. Let G be the orientation obtained from G by orienting the edges of G using the same rules as in Theorem 1. Then the average outdegree in G is (k-1)/2 = (n-2)/4. The result now follows as in Theorem 1.

However this bound is not very tight. For all polygons P we have tried, we have been able to find two opposite vertices such that any circle containing them contains at least half of the vertices of P. This suggests the following problem:

> **Problem 1:** Show that in any convex polygon P with n vertices, there exist two opposite vertices u and v such that any circle containing them contains at least half of the vertices of P.

Final Remarks

An interesting problem is to study how tight the bounds proved in Theorems 1, 2 and 3 are. For example, for small values of n (namely $n < 60$), the results proved in Theorem 1 are far from optimal. In fact, if P_n contains five points, we can always find two elements of P_n such that any circle containing them contains at least a third point in P_n. However, for large values of n, the bound of Theorem 3 may not be far from optimal. Blazek and Koman [1] and Guy [9] have shown that there exist collections of points P_n such that $I(P_n) \leq {}^1/_4 \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor \lfloor (n-2)/2 \rfloor \lfloor (n-3)/2 \rfloor$ which is approximately three times as large as Bin(n,5)/(n-4). This suggests that for some collections, we could have $A(P_n)$ less than or equal to n/20.

We should also mention that Theorem 1 is not valid if P_n is a collection of points in the 2d-dimensional space $\mathbb{R}^{2d}$.

Let $ß(t) = (t, t^2, \ldots, t^{2d})$ and $P_n = \{ß(t_i);\ 0 < t_1 < t_2 < \ldots < t_n\}$. This curve, called the momentum curve, was first discovered by Carathéodory [3], [4]. In [8] (pages 61-62) the following result is proved:

> **Theorem 4:** For any k-pointed subset A of P_n, $k \leq d$ there exists a supporting hyperplane H of the convex hull of P_n such that $H \cap P_n = A$.

As a consequence of this result, we can easily prove the following:

> **Theorem 5:** For any subset A of P_n with $|A| \leq d$, there exists a 2d-dimensional sphere S(A) containing all elements of A such that $S(A) \cap P_n = A$.
>
> This suggests the following problem:
>
> **Problem 2:** Find f(d) and g(d) such that the following holds:

For any collection P_n of n points in $\mathbb{R}^d$ there exists a subset **S** of P_n with f(d) points such that any d-sphere containing **S** contains at least n/g(d) elements in P_n.

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