Ramsey numbers for empty convex polygons

Crevel Bautista-Santiago Javier Cano* Ruy Fabila-Monroy† Carlos Hidalgo Toscano‡
Clemens Huemer§ Jesús Leaños¶ Toshinori Sakai‖ Jorge Urrutia**

Abstract

We study a geometric Ramsey type problem where the vertices of the complete graph K_n are placed on a set S of n points in general position in the plane, and edges are drawn as straight-line segments. We define the empty convex polygon Ramsey number $R_{EC}(k, k)$ as the smallest number n such that for every set S of n points and for every two-coloring of the edges of K_n drawn on S, at least one color class contains an empty convex k-gon. A polygon is empty if it contains no points from S in its interior. We prove $17 \leq R_{EC}(3, 3) \leq 463$ and $57 \leq R_{EC}(4, 4)$. Further, there are three-colorings of the edges of K_n (drawn on a set S) without empty monochromatic triangles. A related Ramsey number for islands in point sets is also studied.

1 Introduction

Ramsey's theorem ensures that for every two-coloring of the edges of the complete graph K_n on a large enough number n of vertices, at least one of the two color classes contains a clique of a given size. The Ramsey number $R(s, t)$ is the smallest number n such that every two-coloring of the edges of K_n contains a clique on s vertices from the first color class or a clique on t vertices from the other color class. Geometric variants of Ramsey's theorem have been studied, see e.g. [9]. When the vertices of K_n are drawn on a set of n points in the plane, and edges as straight-line segments, geometry comes into play by considering crossings of edges. Throughout, we only consider point sets S in general position, meaning sets without three collinear points. For example, in [11] it was shown that for every set S of n points and for every two-coloring of the edges of K_n drawn on S, one color class has non-crossing cycles of lengths $3, 4, \ldots, \left\lfloor \sqrt{n/2} \right\rfloor$. In this work we consider another geometric constraint, namely emptyness. A simple polygon is *empty* if it has no points of S in its interior. The number of empty convex polygons in K_n drawn on sets S of n points have been estimated, see e.g. [1, 2, 7, 10]. We define the empty convex polygon Ramsey number $R_{EC}(s, t)$ as the smallest number n such that for every set S of n points and for every two-coloring of the edges of K_n drawn on S, the first color class contains an empty convex s-gon or the second color class contains an empty convex t-gon. For the case of empty triangles, the bounds $17 \leq R_{EC}(3, 3) \leq 463$ are shown. We also prove that there are three-colorings of the edges of K_n, drawn on some point set S, without empty monochromatic triangles; in other words $R_{EC}(3, 3, 3) = 0$. For the case of empty convex quadrilaterals we can show the lower bound $R_{EC}(4, 4) \geq 57$. We were not able to prove an upper bound. Finally we consider a Ramsey number for islands in point sets. An island of a point set S is a subset I of S such that $Conv(I) \cap S = I$. Islands in point sets were also studied in [3, 4, 6]. In our context, an island is a clique formed by a subset of vertices of K_n drawn on S which contains no further point of S in its interior. We remark that the Ramsey number $R(s, t)$ equals the smallest number n such that every two-coloring of the edges of K_n drawn on a set of n points in convex position contains an island on s points in one color class or an island on t points in the other color class. This is, because there, all islands are in convex position. In [13] it was shown that for every set S of n points, the edges of K_n, drawn on S, can be two-colored such that there is no monochromatic island on four points with triangular convex hull. We prove that there are point sets S and a two-coloring of the edges of K_n, drawn on S, such that there is no monochromatic island on four points (regardless of the form of the convex hull). That is, the island Ramsey number for four points $R_I(4, 4)$ is zero.

*Universidad Nacional Autónoma de México.
†Departamento de Matemáticas, Cinvestav, D.F. México. Partially supported by Conacyt of Mexico, grant 153984.
‡Departamento de Matemáticas, Cinvestav, D.F. México. Partially supported by Conacyt of Mexico, grant 153984.
§Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, BarcelonaTech, Spain. Research supported by projects Gen. Cat. DGR 2014SGR46 and MINECO MTM2012-30951.
¶Unidad Académica de Matemáticas, UAZ, México. Partially supported by Conacyt of Mexico, grant 179867.
‖Department of Mathematics, School of Science, Tokai University, Japan. Research supported by JSPS KAKENHI Grant Number 24540144.
**Instituto de Matemáticas, Universidad Nacional Autónoma de México.

2 The empty triangle Ramsey number

Theorem 1 *The empty triangle Ramsey number satisfies* $17 \leq R_{EC}(3,3) \leq 463$.

Proof. For the upper bound, we use the fact that every sufficiently large point set in general position contains an empty convex hexagon [8, 14]. Koshelev obtained the current best bound, 463, on the number of points needed to guarantee such an empty convex hexagon [12]. Consider only the complete graph on six vertices K_6 formed by the vertices of this hexagon. Ramsey's theorem tells us that every two-coloring of K_6 contains a monochromatic triangle. Since the hexagon is empty, the monochromatic triangle is so as well. For the lower bound, a two-colored complete geometric graph on 16 vertices without an empty monochromatic triangle is shown in Figure 1.

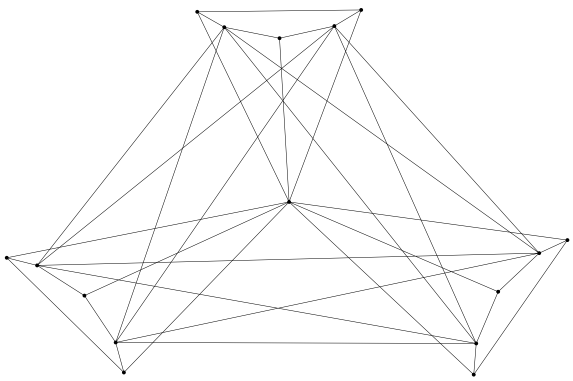

Figure 1: A two-coloring of the edges of K_{16} without an empty monochromatic triangle. Only the edges of one color class are drawn.

□

Theorem 2 *The empty triangle Ramsey number for three-colored complete graphs* $R_{EC}(3,3,3)$ *is zero.*

Proof. We have to present a three-coloring of the edges of the complete geometric graph K_n drawn on a set S of n points. The point set S is the so-called *Horton set* $H(n)$, see e.g. [1, 2, 5, 10], defined recursively as follows: $H(1) = \{(1,1)\}$ and $H(2) = \{(1,1),(2,2)\}$. When $H(n)$ is defined, set

$$H(2n) = \{(2x-1, y) \mid (x,y) \in H(n)\}$$
$$\cup \{(2x, y+3^n) \mid (x,y) \in H(n)\}.$$

In this construction $H(2n)$ is obtained by taking $H(n)$ and a copy of $H(n)$ which is slightly shifted to the right and placed far above the other set $H(n)$. To define an edge-coloring of the complete graph drawn on $H(n)$ we use an auxiliary three-coloring of the vertices of $H(n)$: vertex (x,y) gets color $x \mod 3$. This three-coloring for $H(8)$ is shown in Figure 2. In [5],

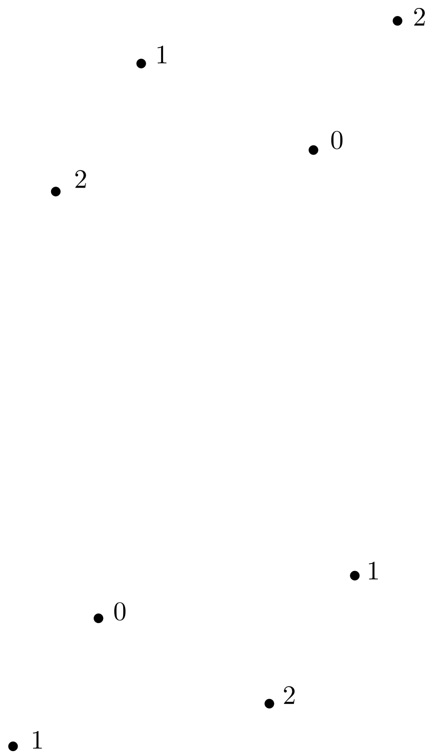

Figure 2: A three-coloring of the vertices of the Horton set $H(8)$.

Theorem 3.3, it was proved that this coloring admits no empty triangles with its three vertices from the same color class. The three-coloring for the edges of K_n is now defined as follows: an edge connecting points (x_1,y_1) and (x_2,y_2) gets color $x_1 + x_2 \mod 3$. Then, a triangle formed by points (x_1,y_1), (x_2,y_2) and (x_3,y_3) is monochromatic if and only if x_1, x_2 and x_3 belong to the same congruence class modulo three. Thus, the vertices of a monochromatic triangle have the same color and from [5] we know that these triangles are not empty. □

3 The empty convex quadrilateral Ramsey number

Theorem 3 *The empty convex quadrilateral Ramsey number satisfies* $57 \leq R_{EC}(4,4)$.

Proof. Figure 3 shows a two-coloring of the edges of K_{11} in convex position without an empty convex monochromatic quadrilateral. A drawing of K_{56} (indicated in Figure 4) and a two-coloring of its edges without an empty convex monochromatic quadrilateral is obtained by placing five groups of 11 points (with two-coloring as in Figure 3) in such a way that the 55 points lie on five small semi-circles with centers the vertices of a regular pentagon. Then the last point is placed in the center of this pentagon and connected to the 55 points with the same color as the drawn

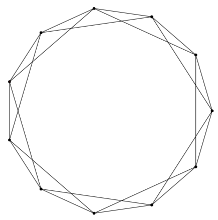

Figure 3: A two-coloring of the edges of K_{11} without an empty convex monochromatic quadrilateral. Only the edges of one color class are drawn.

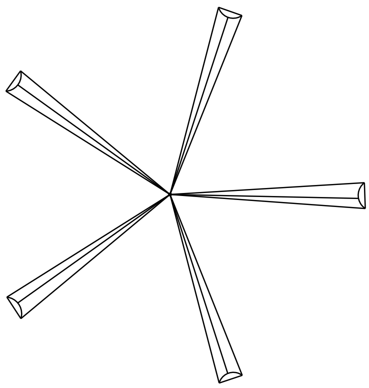

Figure 4: Schematic drawing of K_{56} without an empty convex monochromatic quadrilateral. Only the edges of one color class are indicated.

edges in Figure 3.

□

4 The Ramsey number for islands

Theorem 4 *The island Ramsey number $R_I(4,4)$ is zero.*

Proof. We present a two-coloring of the edges of K_n drawn on the Horton set $H(n)$ without an empty monochromatic K_4. As in the proof of Theorem 2, we start with the auxiliary three-coloring of the vertices of $H(n)$ where vertex (x,y) gets color $x \mod 3$. Now we define a two-coloring for the edges of K_n as follows: an edge connecting points (x_1,y_1) and (x_2,y_2) gets color 0 if $x_1 - x_2 \mod 3 = 0$ and gets color 1 otherwise. In other words, an edge gets color 0 if and only if its two vertices have the same color in the auxiliary vertex coloring. Then, a complete subgraph K_4 is monochromatic if and only if its four vertices have the same color in the auxiliary vertex coloring. Thus, if a K_4 is monochromatic, then from [5] Theorem 3.3, we know that none of its triangles is empty, which implies that this K_4 is not an island.

□

5 Concluding Remarks

An obvious problem left open is to close the gap between lower and upper bound for $R_{EC}(3,3)$. Very interesting would be to prove an upper bound on the empty convex quadrilateral Ramsey number. Computer experiments suggest that it is finite and probably not too large.

References

[1] I. Bárány, Z. Füredi. *Empty simplices in Euclidean space.* Canad. Math. Bull., 30 (1987), 436–445.

[2] I. Bárány, P. Valtr. *Planar point sets with a small number of empty convex polygons.* Stud. Sci. Math. Hung., 41 (2004), 243–269.

[3] C. Bautista-Santiago, J. Cano, R. Fabila Monroy, D. Flores-Peñaloza, H. González-Aguilar, D. Lara, E. Sarmiento, J. Urrutia. *On the connectedness and diameter of a geometric Johnson Graph.* Discrete Mathematics & Theoretical Computer Science 15 (2013), 21–30.

[4] C. Bautista-Santiago, J.M. Díaz-Báñez, D. Lara, P. Pérez-Lantero, J. Urrutia, I. Ventura. *Computing Optimal Islands.* Operations Research Letters, 39 (2011), 246–251.

[5] O. Devillers, F. Hurtado, G. Kàrolyi, C. Seara. *Chromatic variants of the Erdős-Szekeres Theorem.* Comput. Geom., 26 (2003) 193–208.

[6] R. Fabila-Monroy, C. Huemer. *Covering islands in plane point sets.* Lecture Notes in Computer Science, 7579 (2012), 220–225.

[7] A. García. *A note on the number of empty triangles.* Lecture Notes in Computer Science, 7579 (2012), 249–257.

[8] T. Gerken. *Empty convex hexagons in planar point sets.* Discrete and Computational Geometry 39 (2008), 239-272.

[9] H. Harborth, H. Lefmann. *Coloring arcs of convex sets.* Discrete Mathematics, 220 (2000), 107–117.

[10] J.D. Horton. *Sets with no empty convex 7-gons.* Canad. Math. Bull., 26 (1983), 482–484.

[11] G. Kàrolyi, J. Pach, G. Tóth, P. Valtr. *Ramsey-Type results for geometric graphs II.* Discrete and Computational Geometry, 20 (1998), 375-388.

[12] V.A. Koshelev. *On Erdős-Szekeres problem for empty hexagons in the plane.* Model. Anal. Inform. Sist., 16 (2009), 22-74.

[13] J. Nešetřil, J. Solymosi, P. Valtr. *Induced monochromatic subconfigurations.* In: Contemporary Trends in Discrete Mathematics, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 49, AMS, (1999), 219–227.

[14] C.M. Nicolás. *The empty hexagon theorem.* Discrete and Computational Geometry, 38 (2007), 389–397.