Non-crossing Monotonic Paths in Labeled Point Sets on the Plane

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Abstract

Let n be a positive integer, and let P be a set of n points in general position on the plane with labels $1, 2, \ldots, n$. The label of each $p \in P$ will be denoted by $\ell(p)$. A polygonal line connecting k elements $p_1, p_2, \ldots, p_k$ of P in this order is called a *monotonic path of length* k if the sequence $\ell(p_1), \ell(p_2), \ldots, \ell(p_k)$ is monotonically increasing or decreasing in this order. We show that P contains a vertex set of a *non-crossing* monotonic path of length at least $c(\sqrt{n} - 1)$, where $c = 1.0045\ldots$.

1 Introduction

Let P be a set of points on the plane. P is *in general position* if no three of its elements are collinear. Furthermore, P is *in convex position* if all points are vertices of the convex hull of P. All point sets P considered in this paper are in general position, and consisting of points with pairwise different labels $1, 2, \ldots, |P|$. We will refer to these point sets as *lp*-sets. For each *lp*-set P, the label of a point $p \in P$ will be denoted by $\ell(p)$.

Let P be an *lp*-set. A polygonal line connecting k elements $p_1, \ldots, p_k$ of P in this order is called a *monotonic path of length* k if the sequence $\ell(p_1), \ldots, \ell(p_k)$ is monotonically increasing or decreasing (Figure 1). When P contains the vertex set of a non-crossing monotonic path of length k, we will say that P *contains* a non-crossing monotonic path of length k.

The *length* of a finite sequence is the number of its terms. The following theorem is (a corollary of) a well known result by Erdős and Szekeres [3]:

Theorem 1 *Let n be a positive integer. Then any sequence of n distinct real numbers contains a monotonically increasing or decreasing subsequence of length at least $\sqrt{n}$. This bound is tight.*

In [4], Sakai and Urrutia proved that any n-element *lp*-set in *convex* position contains a non-crossing monotonic path of length at least $\sqrt{3n - \frac{3}{4}} - \frac{1}{2}$, improving on a result by Czyzowicz, Kranakis, Krizanc and Urrutia [2]. In [4], it is also conjectured that any n-element *lp*-set in convex position contains a non-crossing monotonic path of length at least $2\sqrt{n} - 1$.

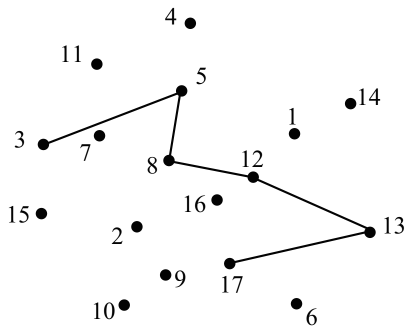

Figure 1: An *lp*-set (each number represents the label of each element) and a monotonic path of length 6.

Furthermore, it has been believed that there exists a constant $c > 1$ such that the following statement holds: any n-element *lp*-set in *general* position contains a non-crossing monotonic path of length at least $c\sqrt{n} - o(\sqrt{n})$. In Section 2, we show the following result:

Theorem 2 *Let n be a positive integer. Then any n-element lp-set P in general position contains a non-crossing monotonic path of length at least $c(\sqrt{n} - 1)$, where* $c = \frac{1}{2}\left(\sqrt{\sqrt{\frac{10}{3}} - 1} + \frac{1}{\sqrt{\sqrt{\frac{10}{3}} - 1}}\right) = 1.0045\ldots$.

Note that it is easy to verify that any n-element *lp*-set contains a non-crossing monotonic path of length at least $\sqrt{n}$. Actually, we have only to take a straight line l that is not perpendicular to any straight line connecting two distinct elements of P, to project all elements of P orthogonally to l, and to apply Theorem 1 to the sequence obtained on l. Though the constant $c = 1.0045\ldots$ in Theorem 2 is just slightly greater than 1, the result shows that the behavior of problems on monotonic sequences and non-crossing monotonic paths are essentially different.

2 Proof of Theorem 2

In this section, we prove Theorem 2 (for $n \geq 4$).

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A finite sequence $\{x_i\}_{i=1}^n$ is said to be *unimodal* (resp. *anti-unimodal*) if there is an m, $1 \le m \le n$, such that $x_1 < x_2 < \cdots < x_m$ and $x_m > x_{m+1} > \cdots > x_n$ (resp. $x_1 > x_2 > \cdots > x_m$ and $x_m < x_{m+1} < \cdots < x_n$). To prove Theorem 2, we use the following Theorem 3 which was first obtained by Chung [1], and later by Sakai and Urrutia [4].

Theorem 3 *Let n be a positive integer. Then any sequence of n distinct real numbers contains a unimodal or anti-unimodal subsequence of length at least* $\sqrt{3n - \frac{3}{4}} - \frac{1}{2}$.

In Figure 2, for $k = 3$, we present an example with $n = 3k^2 + k = 30$ terms whose longest unimodal/anti-unimodal subsequence has length $\left\lceil \sqrt{3n - \frac{3}{4}} - \frac{1}{2} \right\rceil = 3k = 9$.

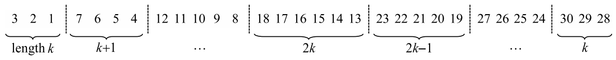

Figure 2: The maximum length of a unimodal/anti-unimodal subsequence is $3k$.

Now we proceed to the proof of Theorem 2. We may assume that P is an lp-set on $\mathbb{R}^2$, and that no two points of P have the same x-coordinate. Let $p_1, p_2, \ldots, p_n$ be the elements of P in increasing order of their x-coordinates, and let $\mathcal{L}$ be the sequence $\ell(p_1), \ell(p_2), \ldots, \ell(p_n)$ (recall that $\ell(x)$ denotes the label of point x), which is a permutation of $\{1, 2, \ldots, n\}$. For each i with $1 \le i \le n$, let a_i denote the length of the longest *increasing* subsequences of $\mathcal{L}$ *ending* at $\ell(p_i)$, b_i the length of the longest *decreasing* subsequences of $\mathcal{L}$ ending at $\ell(p_i)$, and A_i the point (a_i, b_i) on the ab-coordinate plane. Set $\mathcal{A} = \{A_i : 1 \le i \le n\}$. We can verify that the following lemma holds:

Lemma 4 *Let i and j be integers with $1 \le i < j \le n$. Then the following (i) and (ii) hold.*
(i) If $\ell(p_i) < \ell(p_j)$, then $a_j \ge a_i + 1$.
(ii) If $\ell(p_i) > \ell(p_j)$, then $b_j \ge b_i + 1$.

So, for distinct indices i and j, we must have $A_i \ne A_j$.

First consider the case where there exists m such that $a_m \ge c(\sqrt{n} - 1)$ (recall that $c = 1.0045\ldots$, as in the statement of Theorem 2). In this case, there exists a non-crossing path connecting a_m points of P and ending at p_m such that the values of the labels of its vertices monotonically increase along it, as desired. Also, in the case where there exists m such that $b_m \ge c(\sqrt{n} - 1)$, we can find a path with desired properties as well. Thus we may assume that

$$\left.\begin{array}{l} a_i < c(\sqrt{n} - 1) \text{ and } b_i < c(\sqrt{n} - 1) \\ \text{for all } 1 \le i \le n. \end{array}\right\} \quad (1)$$

A Non-crossing Monotonic Path $\mathcal{P}$

Let $d = \sqrt{\sqrt{\frac{10}{3}} - 1} = 0.9087\ldots$. We have $c = \frac{1}{2}\left(d + \frac{1}{d}\right)$, and hence

$$2cd = d^2 + 1. \quad (2)$$

We can also verify the following (3) and (4).

$$0.09 < c - d < 0.1. \quad (3)$$

$$14c^2 - 5d^2 = 10. \quad (4)$$

Lemma 5 *There exists m such that*

$$a_m > d(\sqrt{n} - 1) \text{ and } b_m > d(\sqrt{n} - 1) \quad (5)$$

(Figure 3).

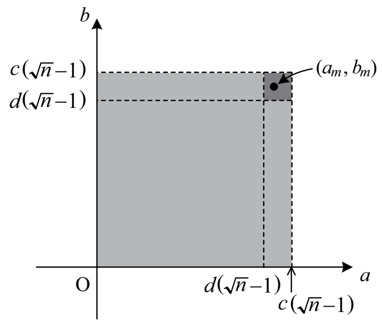

Figure 3

Proof. By way of contradiction, suppose that $a_i \le d(\sqrt{n} - 1)$ or $b_i \le d(\sqrt{n} - 1)$ for all i. From this assumption and (1), it follows that

$$\begin{aligned} |\mathcal{A}| &< [c(\sqrt{n} - 1)]^2 - [(c - d)(\sqrt{n} - 1) - 1]^2 \\ &< n - 2\sqrt{n} + 2(c - d)(\sqrt{n} - 1) \quad \text{(by (2))} \\ &< n \quad \text{(by (3))}, \end{aligned}$$

a contradiction. □

Take m satisfying (5). By symmetry, we may assume that

$$\ell(p_m) \le \frac{n}{2}. \quad (6)$$

Also, by the definition of the a_i, there is a non-crossing path $\mathcal{P}$ connecting a_m points of P and ending at p_m such that the values of the labels of points monotonically increase along $\mathcal{P}$. We have

$$\text{the length of } \mathcal{P} = a_m > d(\sqrt{n} - 1) \quad (7)$$

by (5).

A Path Connecting a Unimodal Sequence

Next we define Q_1 and Q_2 by

$Q_1 = \{p_i : 1 \le i \le m-1 \text{ and } \ell(p_i) > \ell(p_m)\}$, and
$Q_2 = \{p_i : m+1 \le i \le n \text{ and } \ell(p_i) > \ell(p_m)\}$

(so, in particular, the x-coordinates of the elements of Q_1 (resp. Q_2) are smaller (resp. greater) than the x-coordinate of p_m). By Lemma 4 (i) and (5), $a_i \ge a_m + 1 > d(\sqrt{n}-1)+1$ for any $p_i \in Q_2$. From this and (1), it follows that for any $p_i \in Q_2$,

$$\begin{array}{rcccl} d(\sqrt{n}-1)+1 & < & a_i & < & c(\sqrt{n}-1) \quad \text{and} \\ 1 & \le & b_i & < & c(\sqrt{n}-1), \end{array}$$

and hence

$$\begin{aligned} |Q_2| &< (c-d)(\sqrt{n}-1) \times c(\sqrt{n}-1) \\ &= c(c-d)(\sqrt{n}-1)^2. \end{aligned}$$

From this, we obtain

$$\begin{aligned} |Q_1| &= (n - \ell(p_m)) - |Q_2| \\ &> \frac{n}{2} - c(c-d)(\sqrt{n}-1)^2 \\ &> \frac{1}{7}\left(\sqrt{\frac{10}{3}}+1\right)n + \frac{1}{4} \\ &= \frac{1}{3d^2}n + \frac{1}{4} \end{aligned} \tag{8}$$

by (2), (3), (4) and the assumption that $n \ge 4$.

Connect p_m and each element of Q_1, and relabel the elements of Q_1 as $q_1, q_2, \ldots, q_{|Q_1|}$ in the counter-clockwise order around p_m. We choose q_1 in such a way that all other elements of Q_1 lie on the left side of directed line $p_m q_1$.

By Theorem 3 and (8), there exists a path $\mathcal{Q} = q_{i_1} q_{i_2} \ldots q_{i_k}$ of length

$$k \ge \sqrt{3|Q_1| - \frac{3}{4}} - \frac{1}{2} > \frac{1}{d}\sqrt{n} - \frac{1}{2} \tag{9}$$

such that $i_1 < i_2 < \cdots < i_k$, and such that either

(i) $\ell(q_{i_1}) < \cdots < \ell(q_{i_h}) > \ell(q_{i_{h+1}}) > \cdots > \ell(q_{i_k})$ or
(ii) $\ell(q_{i_1}) > \cdots > \ell(q_{i_h}) < \ell(q_{i_{h+1}}) < \cdots < \ell(q_{i_k})$

holds for some h. Define *monotonic* subpaths $\mathcal{R}_1$ and $\mathcal{R}_2$ by

$$\begin{aligned} \mathcal{R}_1 &= q_{i_1} q_{i_2} \ldots q_{i_h} \text{ and} \\ \mathcal{R}_2 &= q_{i_h} q_{i_{h+1}} \ldots q_{i_k}, \end{aligned}$$

and also define ${\mathcal{R}_1}^{-1}$ and ${\mathcal{R}_2}^{-1}$ by

$$\begin{aligned} {\mathcal{R}_1}^{-1} &= q_{i_h} q_{i_{h-1}} \ldots q_{i_1} \text{ and} \\ {\mathcal{R}_2}^{-1} &= q_{i_k} q_{i_{k-1}} \ldots q_{i_h}. \end{aligned}$$

Combining Paths

Let H_1 (resp. H_2) be closed half-plane bounded by straight line $p_m q_{i_h}$ and containing q_{i_1} (resp. q_{i_k}). Let P_0 be the vertex set of $\mathcal{P}$, and write

$$\begin{aligned} P_0 \cap H_1 &= \{p_{j_1}, p_{j_2}, \ldots, p_{j_s}\}, \\ &\quad\ \text{where } j_1 < j_2 < \cdots < j_s, \text{ and} \\ P_0 \cap H_2 &= \{p_{j'_1}, p_{j'_2}, \ldots, p_{j'_t}\}, \\ &\quad\ \text{where } j'_1 < j'_2 < \cdots < j'_t \end{aligned}$$

(note that we have $p_{j_s} = p_{j'_t} = p_m$). Then the paths $\mathcal{P}_1 = p_{j_1} p_{j_2} \ldots p_{j_s}$ and $\mathcal{P}_2 = p_{j'_1} p_{j'_2} \ldots p_{j'_t}$ are non-crossing monotonic paths in H_1 and H_2, respectively (Figure 4).

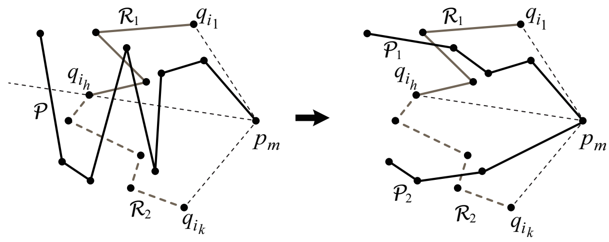

Figure 4

Case 1. $\mathcal{R}_1$ is increasing and $\mathcal{R}_2$ is decreasing:

In this case, we combine paths $\mathcal{P}_1$, $p_{j_s} q_{i_k}$ and ${\mathcal{R}_2}^{-1}$ to form a non-crossing monotonic path $\mathcal{S}_1$, and combine paths $\mathcal{P}_2$, $p_{j'_t} q_{i_1}$ and $\mathcal{R}_1$ to form another non-crossing monotonic path $\mathcal{S}_2$:

$$\begin{aligned} \mathcal{S}_1 &= p_{j_1} p_{j_2} \ldots p_{j_s} q_{i_k} q_{i_{k-1}} \ldots q_{i_h} \quad \text{and} \\ \mathcal{S}_2 &= p_{j'_1} p_{j'_2} \ldots p_{j'_t} q_{i_1} q_{i_2} \ldots q_{i_h}. \end{aligned}$$

Since

$$\begin{aligned} &(\text{the length of } \mathcal{S}_1) + (\text{the length of } \mathcal{S}_2) \\ &\quad = [(\text{the length of } \mathcal{P}) + 1] \\ &\qquad\qquad + [(\text{the length of } \mathcal{Q}) + 1] \\ &\quad = (a_m + 1) + (k+1) \\ &\quad > d(\sqrt{n}-1) + \frac{1}{d}\sqrt{n} + \frac{3}{2} \quad (\text{by (7) and (9)}) \\ &\quad > \left(d + \frac{1}{d}\right)(\sqrt{n}-1), \end{aligned}$$

at least one of $\mathcal{S}_1$ or $\mathcal{S}_2$ has length at least $\frac{1}{2}\left(d + \frac{1}{d}\right)(\sqrt{n}-1) = c(\sqrt{n}-1)$, as desired.

Case 2. $\mathcal{R}_1$ is decreasing and $\mathcal{R}_2$ is increasing:

In this case, we combine paths $\mathcal{P}_1$, $p_{j_s} q_{i_h}$ and $\mathcal{R}_2$ to form a non-crossing monotonic path $\mathcal{T}_1$, and combine paths $\mathcal{P}_2$, $p_{j'_t} q_{i_h}$ and ${\mathcal{R}_1}^{-1}$ to form another non-crossing monotonic path $\mathcal{T}_2$. The rest of the argument is quite similar to the argument in Case 1.

Acknowledgments

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