LIGHT SOURCES, OBSTRUCTIONS AND SPHERICAL ORDERS

by

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Abstract. Ordered sets are used as a computational model for motion planning in which figures on the plane may be moved along a ray emanating from a light source. The resulting obstructions give rise to ordered sets which, in turn, are precisely (truncated) spherical orders. We show too, that there is a linear-time algorithm to recognize such ordered sets.

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This paper is inspired by an article of Rival and Urrutia (1988) in which a computational model for motion planning is introduced based on ordered sets. According to this model, robots are idealized by convex figures on the plane and their motion on the plane is studied by assigning to each a direction along which it may be moved with some velocity. The objective may be to separate these robots efficiently or, perhaps, to relay messages among them.

Let F be a family of closed connected plane figures and x a point on the plane not contained in any element of F. For figures A and B we say that B *obstructs* A (or B *blocks* A) if there exists a point b in B such that the line joining x to b intersects A. We write $A \triangleleft B$. More generally, we write $A<B$ if there is a sequence $A = A_1 \triangleleft A_2 \triangleleft \ldots \triangleleft A_k=B$. This relation $<$ is transitive. We call this binary relation $<$ a *blocking* relation. If the blocking relation has no directed cycles then it is antisymmetric too. In that case the blocking relation $<$ is a (strict) order on the set of these figures. (See Figure 1.)

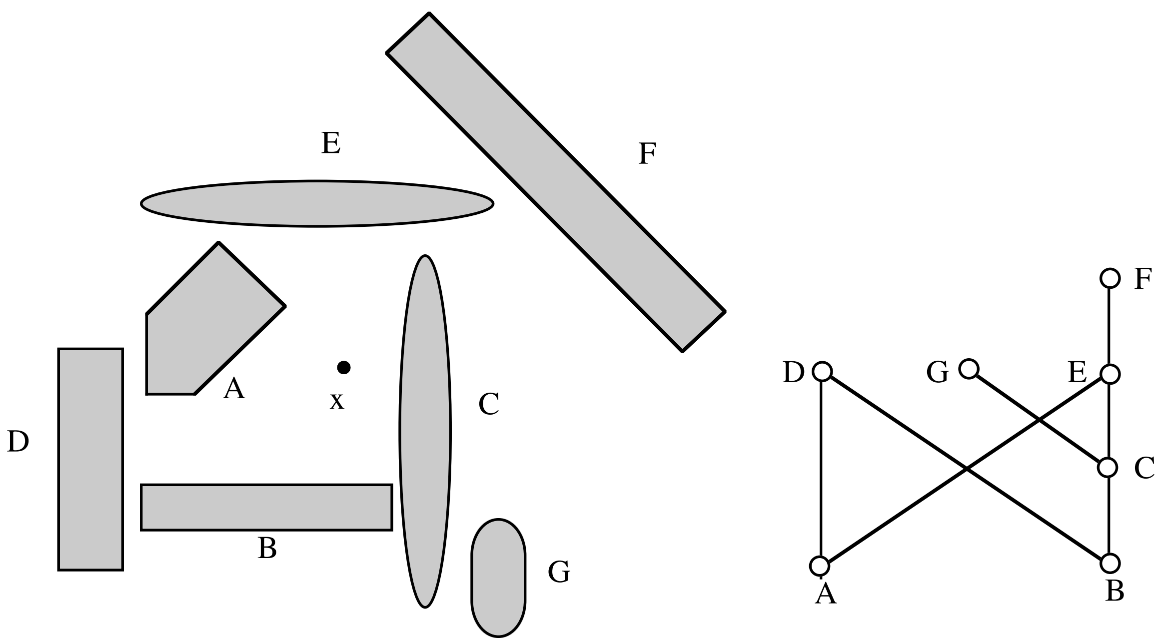


Figure 1

An order P has a *light source* representation if there is a (reference) point x, a set T of pairwise disjoint figures not containing x, and a bijective mapping f of P to T, such that for every $a, b \in P$, $a < b$ if and only if $f(b)$ blocks $f(a)$.

This obstruction notion is a variant of the one-directional blocking relation presented by Rival and Urrutia (1988) for convex figures on the plane. According to them, B is a (one-directional) obstruction of A if some translation of A in the upward vertical direction intersects B.

It is easy to verify that any ordered set representing a one-directional blocking relation also has a light source representation. (See Figure 2.)

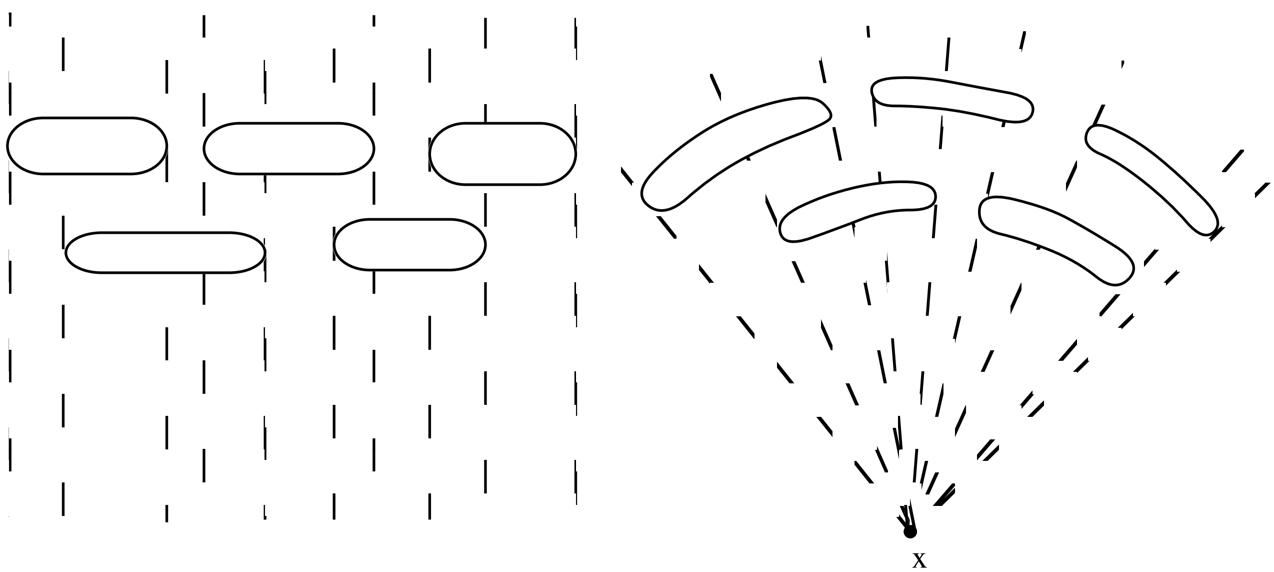


Figure 2

On the other hand, not every order having a light source representation is a one-directional blocking relation. The order in Figure 1, for instance, does not represent a "one-directional blocking relation".

Central to our investigations here is the usual (order) *diagram* of an ordered set. We say that b is an *upper cover* of a, or a is a *lower cover* of b, or a is *covered* by b, if $a < b$ and $a < c \leq b$ imply $b = c$.

The diagram of an ordered set is the directed graph in which $a \to b$ if a is covered by b. The covering graph of an ordered set is the *undirected* graph associated with its diagram. It is convenient and customary to represent the diagram pictorially on the plane with vertices for the elements so arranged that the y-coordinate of a point b is larger than that of another point a if $a < b$ and an edge joins them just if b is an upper cover of a. (See Figure 3.) In this way the arrows on the edges may be ignored.

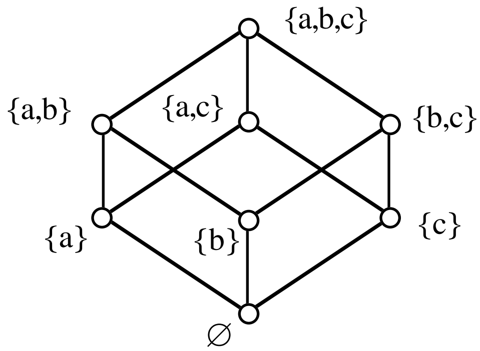
A diagram is p*lanar* if it has a representation on the plane such that

i) if b covers a the y-coordinate of a is smaller than that of b,

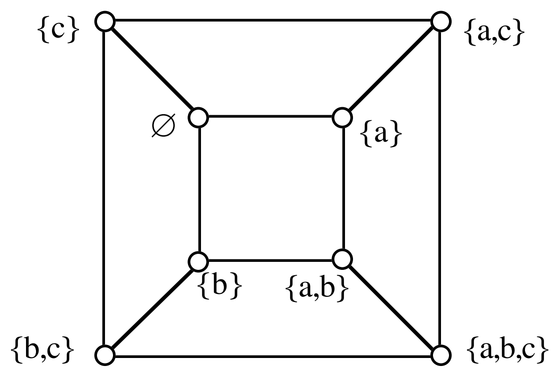
ii) the edge joining a to b is represented by the open straight line segment joining a to b, and

iii) no two edges intersect.

An order is *planar* if its diagram is planar.



The diagram of $\mathbf{2}^3$, the ordered set of all subsets of {a, b, c} ordered by set inclusion.



The covering graph of $\mathbf{2}^3$.

Figure 3

The planarity of the covering graph of an order need not imply planarity of the order itself, for $\mathbf{2}^3$ is a nonplanar ordered set, yet its covering graph certainly is planar (cf. Figure 3).

A *spherical ordered set* (or *spherical order*) is a finite ordered set with bottom and top elements whose diagram can be embedded on the surface of a sphere such that

1) the bottom is mapped to the south pole, the top to the north pole,

2) all arcs are strictly increasing northward, and

3) no pair of arcs cross except at an element of the underlying set.

A *truncated spherical order* is an ordered set obtained from a spherical order by removing its bottom and top.

The ordered set $\mathbf{2}^3$ is a spherical order. On the other hand, the ordered set illustrated in Figure 4 is not. According to the ordered set of Figure 1, a spherical order need not be a lattice yet, in a spherical order, every pair of elements has at most two minimal upper bounds and, at most two maximal lower bounds (cf. Figure 5). (Recall, a *lattice* is an ordered set in which every pair of elements has supremum and infimum.) Still, every lattice whose diagram is planar is a spherical order; not every ordered set with planar diagram is spherical (cf. Figure 4). As a planar order with top and bottom is actually a (planar) lattice it follows, too, that every planar order with top and bottom is spherical (cf. Kelly and Rival [1975]).

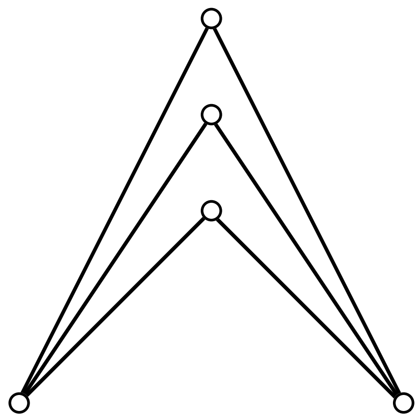


Figure 4

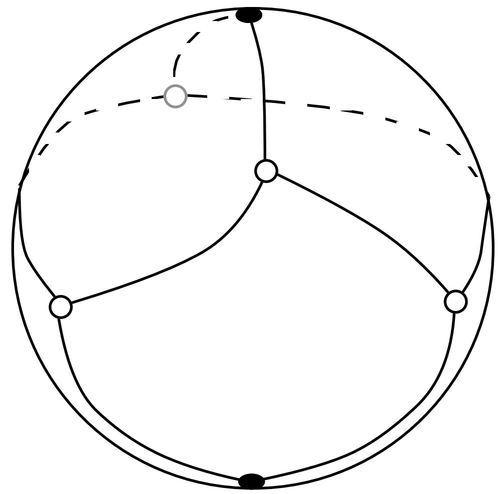
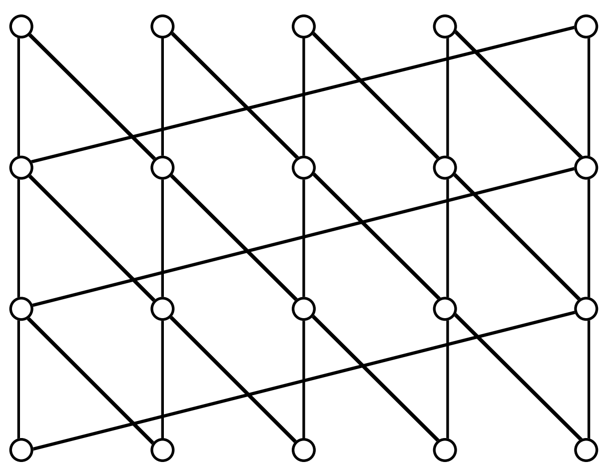


Figure 5

Neither do spherical orders have bounded (order) dimension.
An ordered set constructed by "gluing" n-2 identical copies of n-cycles (as illustrated in Figure 6) has dimension n for it contains the subset of $\mathbf{2}^n$ consisting of its singletons (minimals) and one-element deleted subsets (maximals).



A spherical order with
dimension five
Figure 6

Our principal results show that the theories of orders with a light source, on the one hand, and spherical orders on the other, are really identical.

THEOREM 1. *An ordered set is spherical if and only if it has a bottom, a top, and its covering graph is planar.*

THEOREM 2. *An ordered set has a light source representation if and only if it is truncated spherical.*

THEOREM 3. *There is an* O(n) *algorithm to decide whether an ordered set with* n *elements has a light source representation.*

Proof of Theorem 1

The following lemma is a variant of a result of Platt [1976], and its proof is essentially the same.

LEMMA 1**.** *Let* L *be a lattice,* D *its diagram,* G *its covering graph, and let* h *be a strictly increasing function from* L *to* $\mathbb{R}$*. If* G *is a planar graph and can be drawn on the plane in such a way that the bottom and the top of* L *lie on the same face* F *of* G*, then* L *is a planar lattice and may be represented in the plane with straight line arcs in such a way that*

1) F *is the outer face of the representation and,*

2) every element x *of* L *is represented by a point in* $\mathbb{R}^2$ *whose second coordinate is* h(x).

Here is a sketch of the proof. First, in the planar representation of G, we may assume that F is the outer face. Second, the bottom-to-top paths, bounding this outer face F, correspond to directed paths in the diagram D. Third, if any of these paths is of length greater than one, then it contains an inner vertex that has degree 2 in G. Removing this vertex leaves us with a smaller lattice. An induction completes the proof.

We turn now directly to the proof of Theorem 1.

PROOF. Clearly, the conditions are necessary. Conversely, let D be the diagram of an ordered set satisfying these conditions. Since D, as a graph, is planar, it can be embedded on the sphere. Moreover, it is easy to verify that one such embedding exists in which the bottom (S) and top (N) of D are on the south and north poles of the sphere. (Arcs of the graph are of course, not necessarily mapped with a northward orientation.)

We define a directed graph D' as follows. Let P be any directed path from S to N with vertices $S = x_0, x_1,...,x_n = N$. For each internal vertex x_i a sufficiently small neighborhood of it is divided by P into a "left" and a "right" part. The arcs incident with x_i other than (x_{i-1}, x_i) and $(x_i,$

x_{i+1}) are accordingly classified as being on the left or right of x_i. The vertex set of D' is obtained from that of D by replacing each internal vertex x_i of P by two new distinct vertices x_{iL} and x_{iR}. The arcs of D' consist of all those of D, except those whose head or tail is an internal vertex of P; plus

$$(x_0, x_{1L}), (x_{1L}, x_{2L}), \ldots, (x_{(n-1)L}, x_n),$$
$$(x_0, x_{1R}), (x_{1R}, x_{2R}), \ldots, (x_{(n-1)R}, x_n),$$

plus, for every internal vertex x_i of P and every arc of D incident with x_i on the left or on the right, a new arc defined by

arc of D *incident with* x_i	*on which side*	*new arc*
(y, x_i)	left	(y, x_{iL})
(x_i, y)	left	(x_{iL}, y)
(y, x_i)	right	(y, x_{iR})
(x_i, y)	right	(x_{iR}, y)

We may say that D' is obtained from D by *splitting* the path P into $P_L = (S, x_{1L}, \ldots, x_{(n-1)L}, N)$ and $P_R = (S, x_{1R}, \ldots, x_{(n-1)R}, N)$ (see Figure 7).

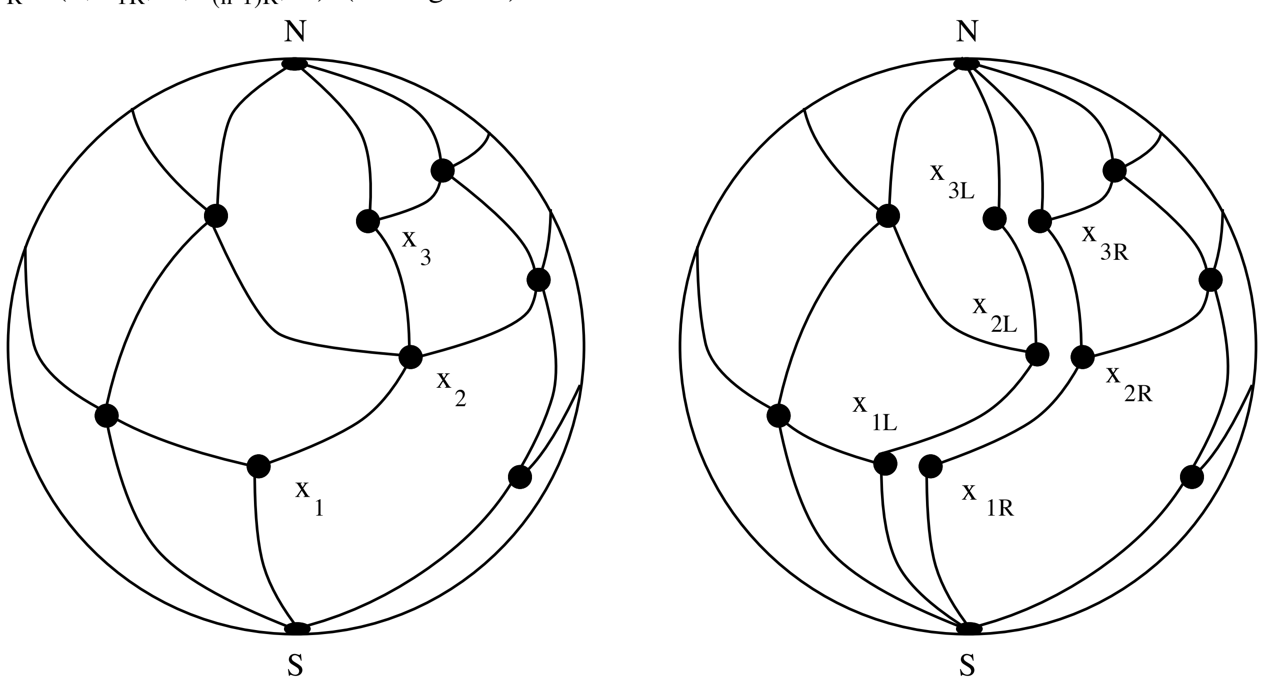


Figure 7

Considered as an undirected graph, D' is planar. A representation of D' (undirected) on the sphere is obtained with P_L and P_R defining a new face from the original representation of D (undirected) on the sphere. Also D' itself represents an order L. Moreover, we shall show that this order L is a lattice. It will then follow, via Lemma 1, that L is a planar lattice.

Since L has bottom $S = x_0$ and top $N = x_n$, to prove that L is a lattice, it suffices to show that no pair a, b of elements of L has either two distinct minimal upper bounds u and v, or two maximal lower bounds. Assuming the contrary, for upper bounds, we would have four distinct directed paths in D', P_{au} from a to u, and similarly, P_{av}, P_{bu}, P_{bv}. Moreover, we may assume that P_{au} and P_{av} have only the vertex a in common—otherwise we would replace a by the last common vertex of the two paths. Similarly, we may assume that P_{bu} and P_{bv} have only b in common.

We fix a planar representation of D' (as undirected graph) with P_L and P_R bounding the outer face. Both S and N are then strictly in the outside region of the simple closed curve J determined by P_{au}, P_{bu} (reversed), P_{bv}, P_{av} (reversed).

Let o be a maximal lower bound of a and b. As no S–o path can meet the curve J, o is also outside J. Let P_{oa} and P_{ob} be directed paths from o to a and b, respectively. P_{oa}, P_{au}, P_{bu} (reversed), P_{ob} (reversed) define a closed curve J_u. Similarly, P_{oa}, P_{av}, P_{bv} (reversed), P_{ob} (reversed) define a closed curve J_v. Obviously either u is inside J_v, or v is inside J_u. By symmetry, we may suppose that u is inside J_v (see Figure 8).

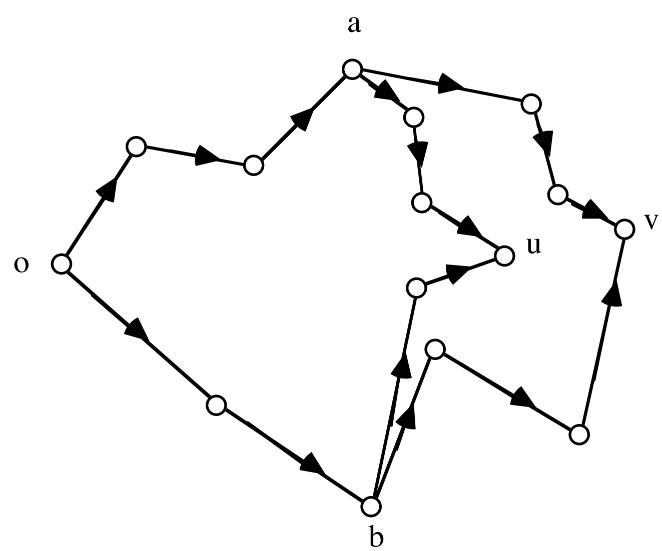


Figure 8

As N lies strictly on the outside of J_v and no directed u–N path can meet J_v, we have a contradiction. The dual argument shows similarly that a and b cannot have distinct maximal common lower bounds, and therefore L is a lattice.

Let C be a linear extension of the ordered set represented by D. For every $x \in L$ let r(x) be the rank of x in the chain C. Define a function h of L to $\mathbb{R}$ as follows. Let h(x) = r(x) if x is not an internal vertex of the split path P. For an internal vertex x_i within P, let $h(x_{iL}) = r(x_i)$ and $h(x_{iR}) = r(x_i)$. Clearly the lattice L, together with the mapping h and face F bounded by P_L and P_R, satisfies the conditions of Lemma 1. The first coordinates of the lattice points in the planar representation of L provided by Lemma 1 can be modified in such a way that

1) all the $x_{1L},\ldots, x_{(n-1)L}$ have the same first coordinate C_L

2) all the $x_{1R},\ldots, x_{(n-1)R}$ have the same first coordinate C_R

3) for every x neither internal in P_L nor internal in P_R, the first coordinate of x is greater than C_L and smaller than C_R.

The entire representation of L then lies between two vertical lines. If the region of $\mathbb{R}^2$ lying between these two lines is mapped in the obvious way to the surface of a cylinder, each internal vertex x_{iL} of P_L is again "identified" with x_{iR} of P_R. In this way we have a representation of the original diagram D on the cylinder, with monotonic arcs. A spherical representation of D on the sphere is now easily obtained.

Proof of Theorem 2

Let P be an order with a light source, represented by a set T of figures, with reference point x and bijection f of P to T. Let u□ v be an arc of the diagram D of P, that is, v covers u in P. Then f(v) obstructs f(u), and there are a □ f(u), b □ f(v) such that a lies on the segment x–b. Let □(u, v) be the point of f(u) lying on the segment x–b that is closest to b. Similarly, let □(u, v) be the point of f(v) lying on the segment x–b that is closest to a. It is easy to see that if (u', v') is another arc of the diagram D, then

{□(u, v), □(u, v)} □ {□(u', v'), □(u', v')} = ∅.

For every u □ P, let

F(u) = {□(u, v): u□ v □ D} □ {□(v, u): v□ u □ D}.

F(u) is a subset of the boundary of f(u). Let u_0 be any point in the interior of f(u). Let u_0 be joined to each point z □ F(u) by a continuous (not necessarily straight) line`through the interior of f(u) in such a manner that lines corresponding to distinct points of F(u) meet only at u_0. (This is possible because f(u) is homeomorphic to a circular disk.) For each z □ F(u), call this continuous line the *link* between u_0 and z. For each arc u□ v of D, let us juxtapose the link from u_0 to □(u, v), the straight line segment from □(u, v) to □(u, v), and the link from □(u, v) to v_0. These juxtaposed lines, linking the various points u_0 for u □ P, form a planar representation of D undirected. (See Figure 9.)

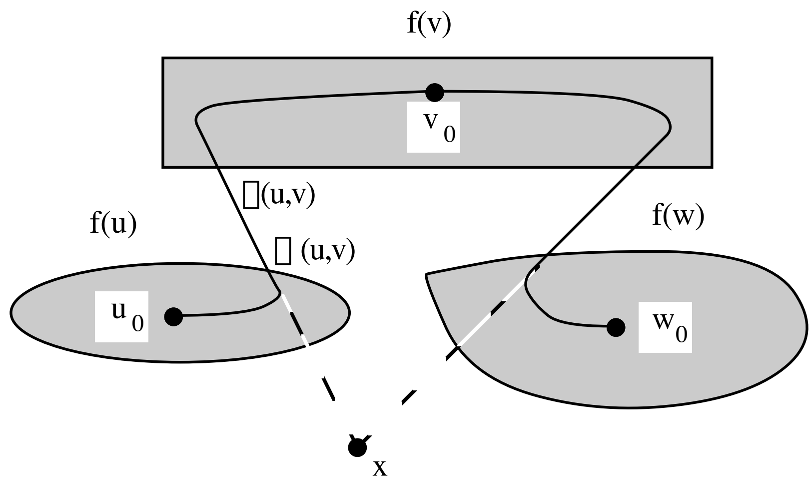


Figure 9

Let y be a point in $\mathbb{R}^2$ whose distance from the reference point x is greater than the distance of any point of any member of S from the reference point x. Define the directed graph ^D by adding x and y to the vertex set of D, an arc $x \to u$ for each element u minimal in P, and an arc $u \to x$ for each element u maximal in P. Clearly ^D is also a diagram, and the planar representation of D undirected, constructed above can be extended to a planar representation of ^D undirected. Furthermore, x is the unique vertex in ^D of indegree zero and y is the unique vertex of outdegree zero. Applying Theorem 1, ^D is a spherical order. It follows that P, represented by D, is truncated spherical.

Conversely, let P be a truncated spherical order. Let ^P be the spherical order obtained from P by adding a (new) bottom S and a (new) top N. Let D be the diagram of ^P. As in the proof of Theorem 1, let D' be the directed graph obtained form D by splitting a directed S–N path

$Q = (S, x_1,\ldots, x_{n-1}, N)$ into

$Q_L = (S, x_{1L},\ldots, x_{(n-1)L}, N)$ and

$Q_R = (S, x_{1R},\ldots, x_{(n-1)R}, N)$.

D' is the diagram of a planar lattice L. $L_0 = L \setminus \{S, N\}$ is a truncated planar lattice. It was shown by Rival and Urrutia [1988] that L_0 can be represented in the plane by a set T of horizontal line segments of finite length, by means of a bijective mapping f of L_0 to T such that, for a, b, $\in L_0$ a $\leq$ b if and only if some upward vertical translation of f(a) intersects f(b). Indeed, their proof of this result indicates that this representation may be made such that the left endpoints of all the segments $f(x_{iL})$, $i = 1,\ldots,n-1$, lie on the same vertical line, and all the right endpoints of all the segments of $f(x_{iR})$, $i = 1,\ldots,n-1$ lie on another vertical line (see Figure 10). Notice that in the representation T obtained for L_0 the line segments representing the x_{iL}'s may be at different "heights" from these representing the x_{iR}'s.

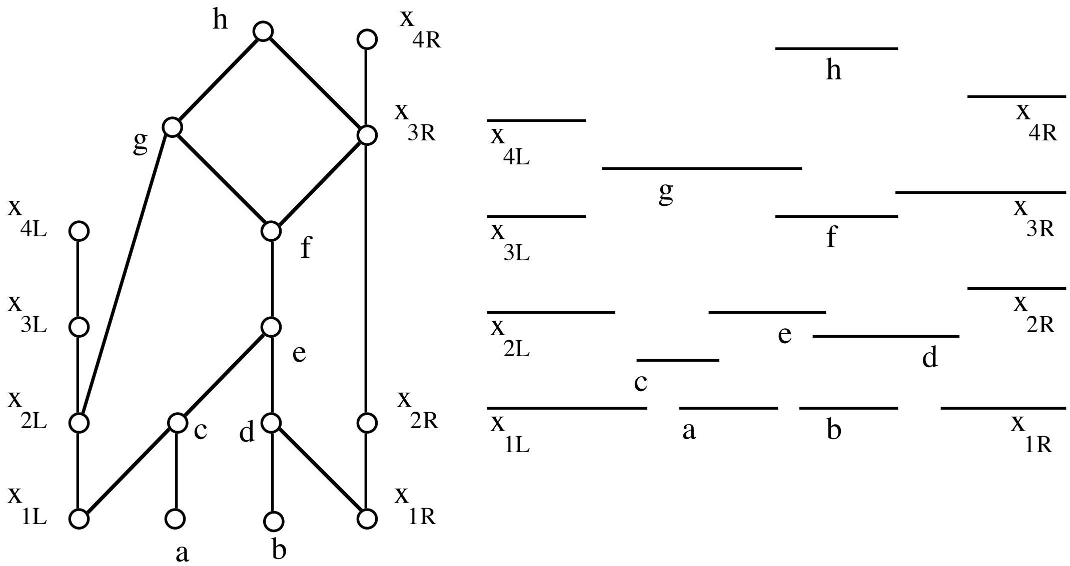


Figure 10

Consider any linear extension C of P and for every element $a \in P$ let $r(a)$ be the rank of a in C. Shift the elements of the representation T of L_0 vertically in such a way that the height of the line segment representing an element $a \in L_0$ is precisely $r(a)$ for elements not in the S–N path Q and the heights of the segments representing x_{iL} and x_{iR} in T are boths $r(x_i)$ (see Figure 11).

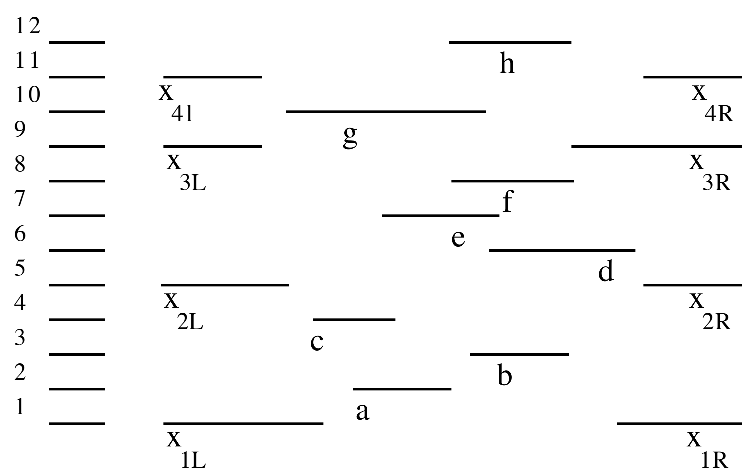


Figure 11

It is easy to see that this shifting in heights still yields a representation of D' although the heights of the line segments representing pairs x_{iL} and x_{iR} are now the same!

If the region of $\mathbb{R}^2$ lying between the two vertical lines bounding the entire representation of L_0 is mapped in the obvious way to the surface of a cylinder, $f(x_{iL})$ and $f(x_{iR})$ become contiguous for each $i = 1,\ldots,n-1$. For every $a \in P \setminus \{x_1,\ldots,x_{n-1}\}$ let $g(a)$ be the line segment $f(a)$ "drawn on the cylinder". For $i = 1,\ldots,n-1$ let $g(x_i)$ be the union of the juxtaposed line segments $f(x_{iL})$ and $f(x_{iR})$ drawn on the cylinder. Clearly, g is injective and, for $a,b \in P$, $a<b$ if and only if, on the cylinder, some upward vertical translation of $g(a)$ intersects $g(b)$.

Now fix a "projection point" s situated on the axis of the cylinder, above all of the $g(a)$, $a \in P$. Also fix a plane orthogonal to the cylinder axis, called the "projection plane", below all of the $g(a)$, $a \in P$. Using s, we can now project each arc $g(a)$ into an arc $p(a)$ on the projection plane. (See Figure 12). Let x be the intersection point of the cylinder axis with the projection plane. It now follows that the set $\{p(a): a \in P\}$ with the point x form a light source representation of P. Our result now follows.

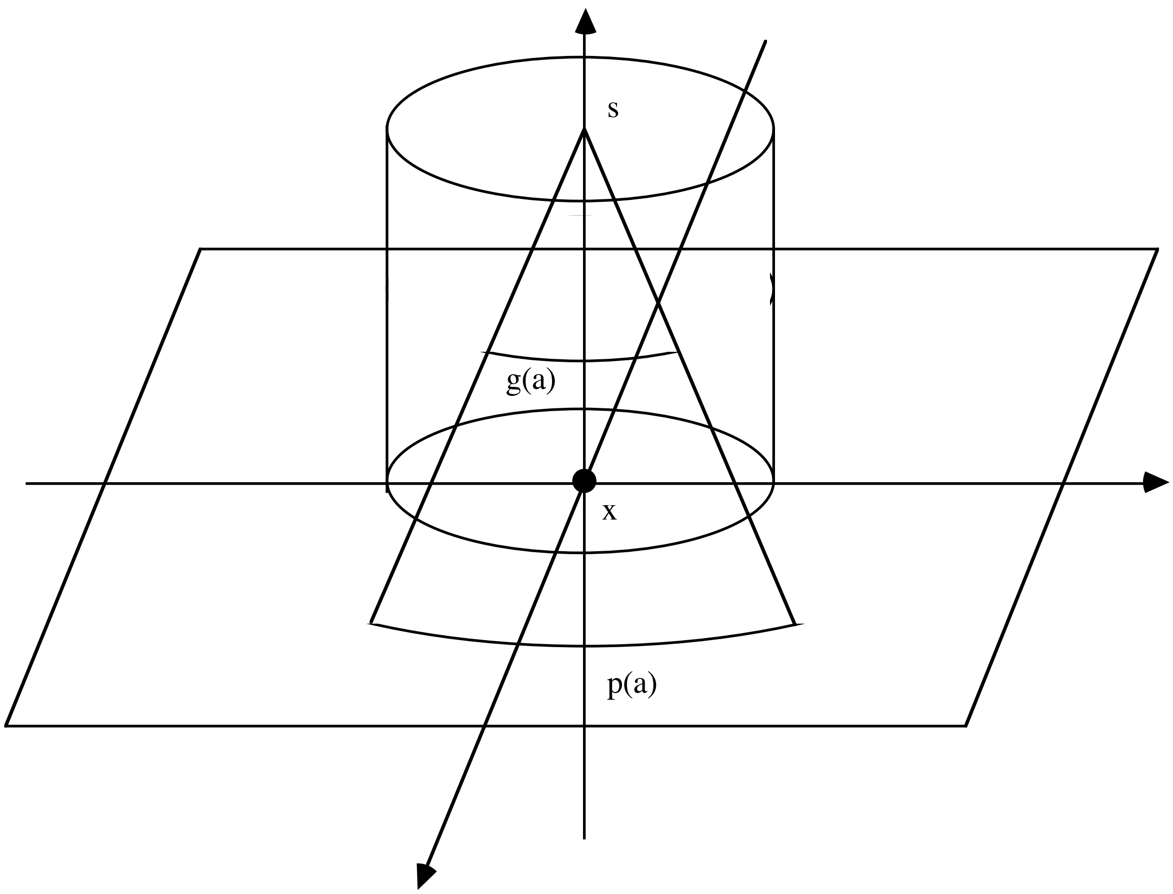


Figure 12

Proof of Theorem 3

Let us suppose that an ordered set with n elements is presented by the incidence matrix of its covering graph. According to Hopcroft and Tarjan [1974] there is an $O(n)$ algorithm to test the planarity of this graph. Then in $O(n)$ time, too, we can locate a minimal element and test whether it is the bottom. Similarly we may test for the top. Therefore, by Theorem 1, we have a linear time algorithm to test whether this order is spherical. By Theorem 2, we then have a linear time algorithm to test whether it has a light source.

In this paper we showed that all truncated spherical orders have a light source representation. For some of these orders, a representation using convex figures is possible. Nevertheless this is not necessarily true for all spherical orders. The order presented in Figure 13 is such that in any light source representation either one of S or N has to be represented using a nonconvex figure.

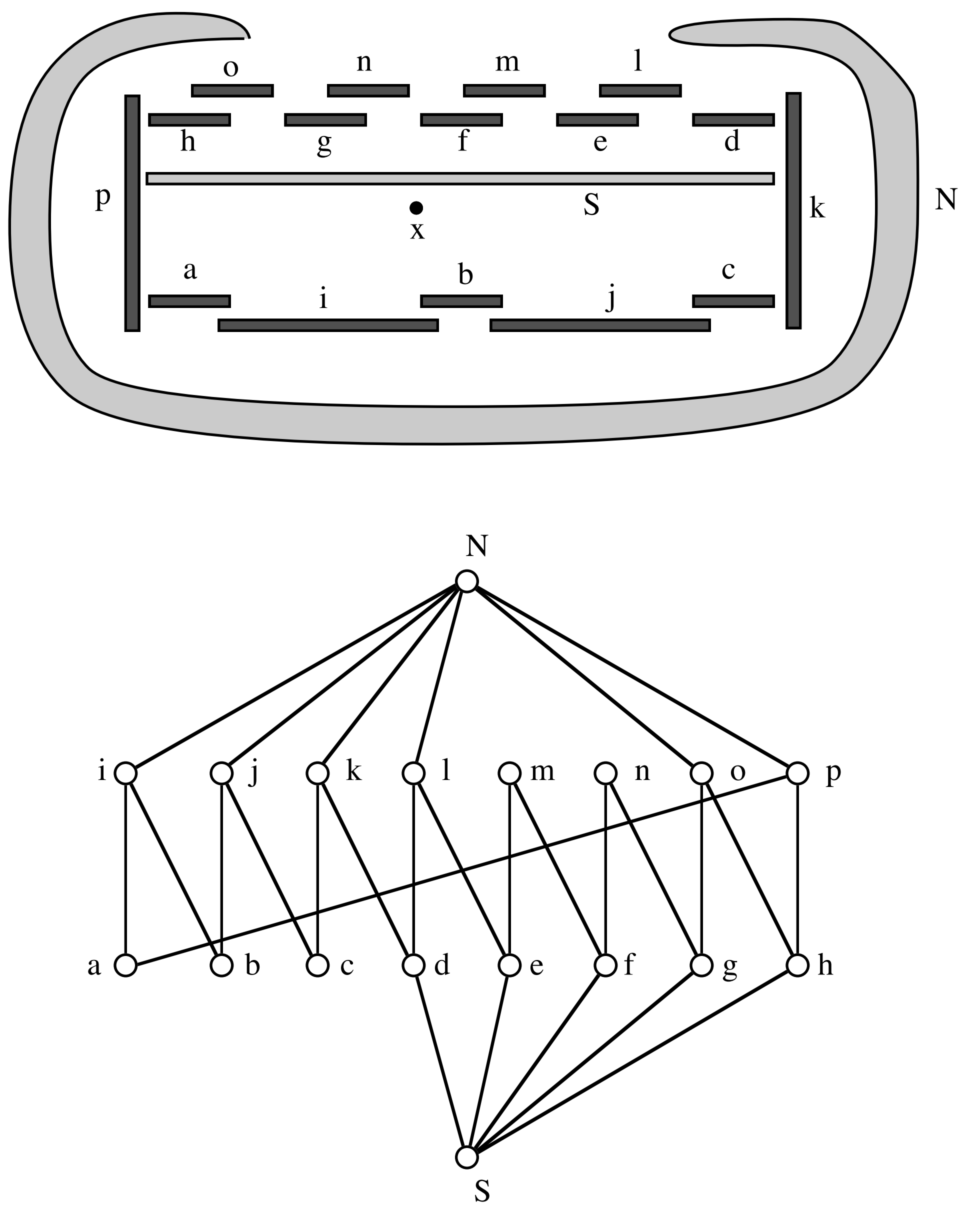


Figure 13

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