REPRESENTING ORDERS ON THE PLANE BY TRANSLATING POINTS AND LINES

by

Richard Nowakowski

Department of Mathematics, Dalhousie University, Halifax, Canada.

Ivan Rival and Jorge Urrutia

Department of Computer Science, University of Ottawa, Ottawa, Canada.

Given a finite collection of disjoint, convex figures in the plane, it is always possible to assign to each a single direction of motion so that this collection of figures may be separated, through an arbitrarily large distance, by translating each figure one at a time, along its assigned direction. Indeed, it is well-known that this separation may be accomplished even if every one of the figures is assigned the same parallel direction (cf. [L. J. Guibas and F. F. Yao (1980)], [I. Rival and J. Urrutia (1987)]). If the convex figures represent robots then the directions of motion may be part of a motion planning scheme to separate the robots without collisions. Or perhaps, the convex figures represent a cluster of figures on a computer screen to be shifted about to clear the screen without altering their integrity and without collisions. These are instances of the problem known in computational geometry as the "separability problem". Rival and Urrutia (1987) have recently initiated the study of this separability problem using a computational model based on the theory of ordered sets.

For figures A and B we say that B *obstructs* A and write A ~ B if there is a line joining a point of A to a point of B which follows the direction assigned to A. We write A<B and say that B *blocks* A if there is a sequence $A=A_1 \sim A_2 \ldots A_k=B$. As long as this blocking relation has no directed cycles then it is a (strict) order on the collection of these figures. Rival and Urrutia (1987) call a collection of disjoint, convex figures (each assigned one of m directions an (m-*directional*) *representation* of an ordered set P if its blocking relation is identical to the ordering of P. Also say that the blocking relation is m-*directional*. They proved these three fundamental results:

F1. *There is a one-to-one correspondence between the class of all one-directional blocking relations and the class of all truncated planar lattices.*

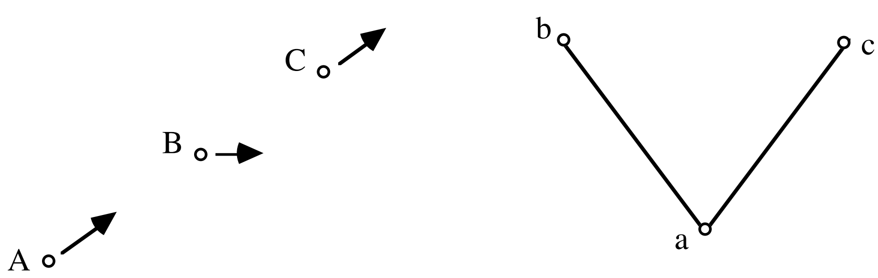
F2. *Every ordered set has a subdivision with a two-directional representatation.*

F3. *There are ordered sets with no* m-*directional representation, for any positive integer* m.

We delineate two directions in the study of the representations of orders as blocking relations:

I. *The convex figures are special (eg. rectangles, line segments, points).*
II. *The number of directions is limited (eg. two, one).*

For example, for every one-directional blocking relation, line segments (perhaps of different lengths) suffice for the convex figures. Moreover, for every positive integer m there are blocking relations requiring m directions. Our aim in this paper is to pursue the first of these two directions in what would seem to be the simplest cases of all, *point blocking relations*, that is, each of the convex figures is a point, and *line blocking relations*, that is, each of the convex figures is a line segment, possibly a point. The most immediate and obvious effect of the restriction to points, for instance, is that if A, B, C are three points on the plane in a blocking relation with A<B, A<C, and B, C are noncomparable then the points A, B and C must be collinear, all lying along the direction assigned to A. Also, at least one of B, C must have a direction not parallel to the direction of A, for otherwise either B<C or C<B (see Figure 1a).



A blocking relation for the ordered set {a<b, a<c}

(a)

An order diagram for the ordered set {a<b, a<c}

(b)

Figure 1

This "collineation constraint" will come to play, as we shall see, an important role.

Such blocking relations must, of course, be carefully distinguished from order diagrams (cf. Figure 1b), which also use points (about which more later).

Here are our results about point blocking relations.

Theorem 1. *There are finite ordered sets with no point blocking representation at all, yet every finite ordered set has a subdivision with a point blocking representation.*

For an ordered set P let P^{top} stand for the ordered set with a top element adjoined, that is, x<top for each x in P; let P_{bottom} stand for the ordered set with a bottom element adjoined, that is, bottom<x for each x in P.

Theorem 2. *If* P *is a finite point blocking relation then* P^{top} *is a point blocking relation too. However, there are finite point blocking relations* P *such that* P_{bottom} *has no point blocking relation at all.*

An interesting consequence is that the dual of a point blocking relation need not be a point blocking relation. This contrasts with the conjecture of Rival and Urrutia (1987) that the dual of a blocking relation is a blocking relation.

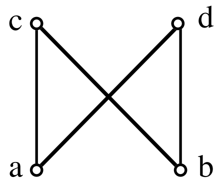
It is surprising that we have as yet no example of a blocking relation which requires convex figures more complicated than points or line segments (cf. **Conjecture II**, in Rival and Urrutia (1987)). Our third result limits the convex figures to line segments .

Theorem 3. *Every series parallel ordered set is a line blocking relation.*

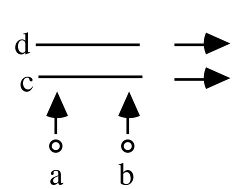
Our final result limits the number of directions to two, as well as the convex figures to lines.

Theorem 4. *Every interval order is a line blocking relation requiring at most two directions.*

There are interval orders which are not point blocking relations and which require at least two directions (see Figure 2).



An interval order

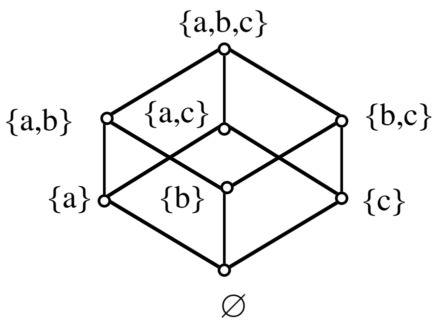


A line-blocking relation

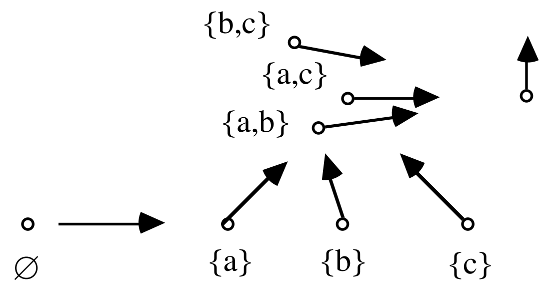
Figure 2

Point Blocking Relations

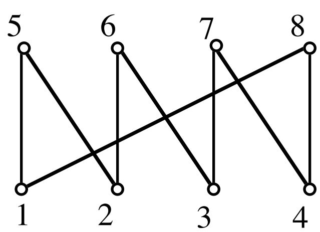
Among the many graphical representation schemes in use to represent ordered sets, one alone is by far the most common, the "order diagram" or, more simply, the "diagram". It is constructed as follows. For an ordered set P and elements a, b in P, we say that b *covers* a or a is *covered* by b, in symbols b >—a or a—< b, if b>a and, for each c in P, b>c≥a implies c=a. We also call b an *upper cover* of a or a *lower cover* of b. If there is a blocking relation that corresponds to the ordered set P, then the upper covers of a convex figure A are those convex figures obstructing A and minimal in the blocking relation with respect to this property. Such upper covers B of A we shall also render with the symbol B>—A or A—<B. It is the custom to represent P pictorially on the plane by means of a *diagram* in which small circles, corresponding to the elements of P, are arranged in such a way that, for a and b in P, the circle corresponding to b is higher than the circle corresponding to a whenever b>a, and a straight line segment connects the two circles whenever b covers a.



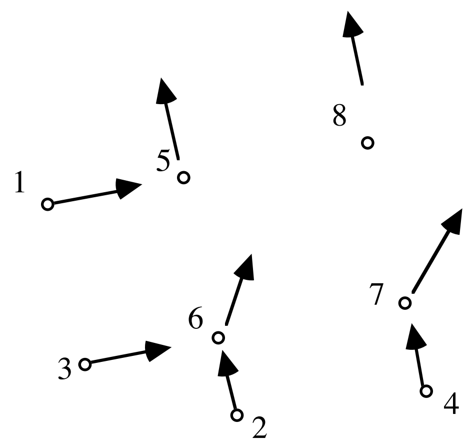
A diagram of $\mathbf{2}^3$, the ordered set of all subsets of {a, b, c} with respect to set inclusion.



A point blocking relation of $\mathbf{2}^3$.



A diagram of a cycle C_8.



A point blocking relation of C_8.

Figure 3

Our aim now is to prove the first part of Theorem 1. We construct an ordered set which cannot be a blocking relation at all. It is convenient to do this in steps. To begin with, we consider the ordered set P with a diagram illustrated in Figure 4. This ordered set P is a point blocking relation. One such point blocking representation of it is illustrated in Figure 5.

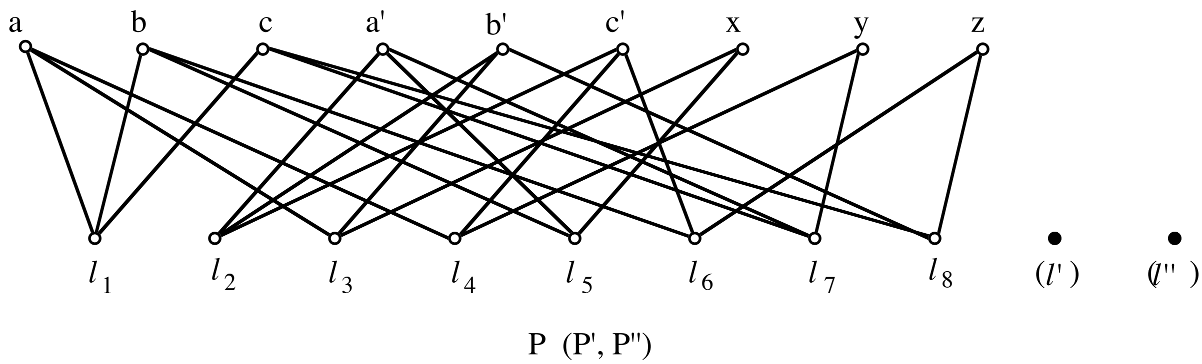


Figure 4

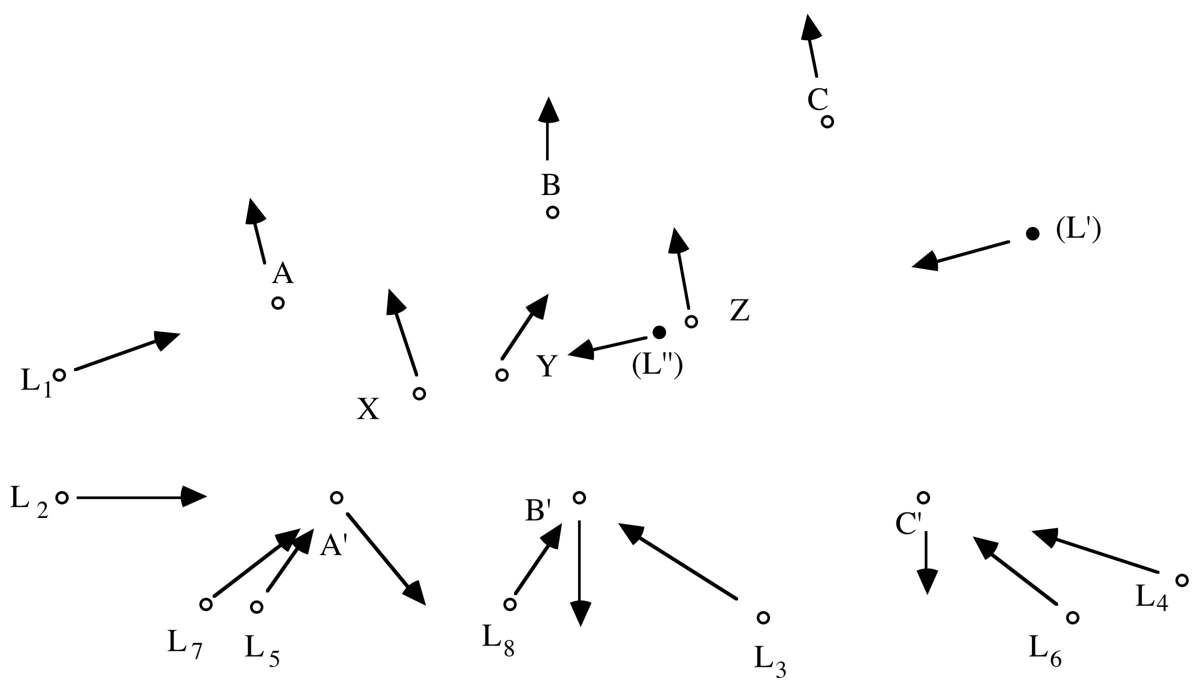


Figure 5

Notice that the direction assigned to a maximal point A, B, C, etc. is arbitrary as long as none is directed at any other point of the collection. Moreover, while there is some variation possible in the serial position of A, B, and A', B', C' and X, Y, Z, for instance, the important point is that A, B, C and A', B', C' are collinear triples so, according to the Pappus Theorem the intersection points X, Y, Z of the corresponding pairs of lines must be collinear.

Thus, the ordered set P' obtained from P by adjoining yet another minimal element l', with the comparabilities l'<x, l'<y, l'<z is a point blocking relation, too, for we may locate a point for L' in Figure 5 collinear with X, Y and Z, say to the right of Z and

directed leftward along the X, Y, Z line. Similarly, the ordered set P" obtained from P by adjoining a minimal element l'' subject only to the comparabilities $l''<x$ and $l''<y$ has a point blocking representation, too, in which a point L" may be put between Y and Z on the line joining them and assigned the direction leftward to Y.

Nevertheless, the ordered set Q illustrated in Figure 6 and obtained from P by adjoining two minimal elements l_9 and l_{10} subject only to $l_9<x$, $l_9<y$, $l_{10}<y$, and $l_{10}<z$ cannot have a point blocking representation. If it did then the points X, Y, Z would, as before, according to the Pappus Theorem, be collinear. Now, both L_9 and L_{10} must lie on this line and each must be assigned a direction along this line coinciding with X, Y and Y, Z, respectively, and, to maintain the noncomparability of l_9 and l_{10}, these directions must be opposite. This is impossible and so Q cannot be a point blocking relation.

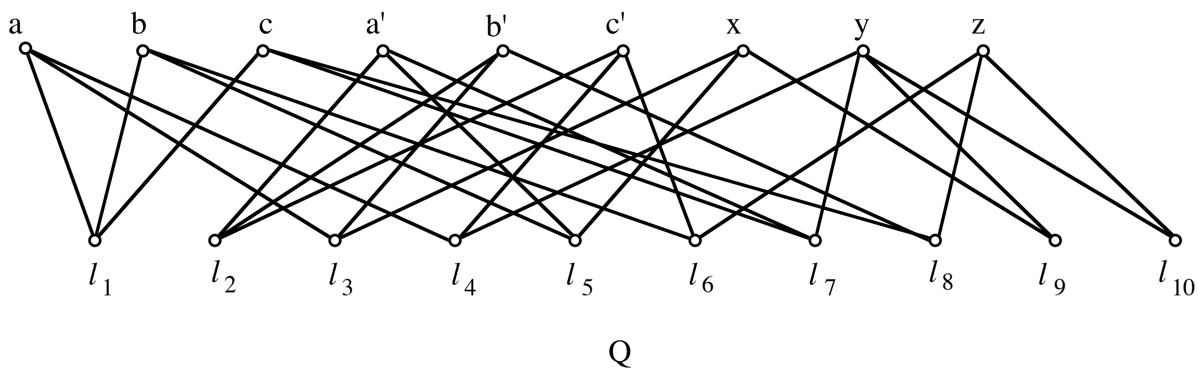
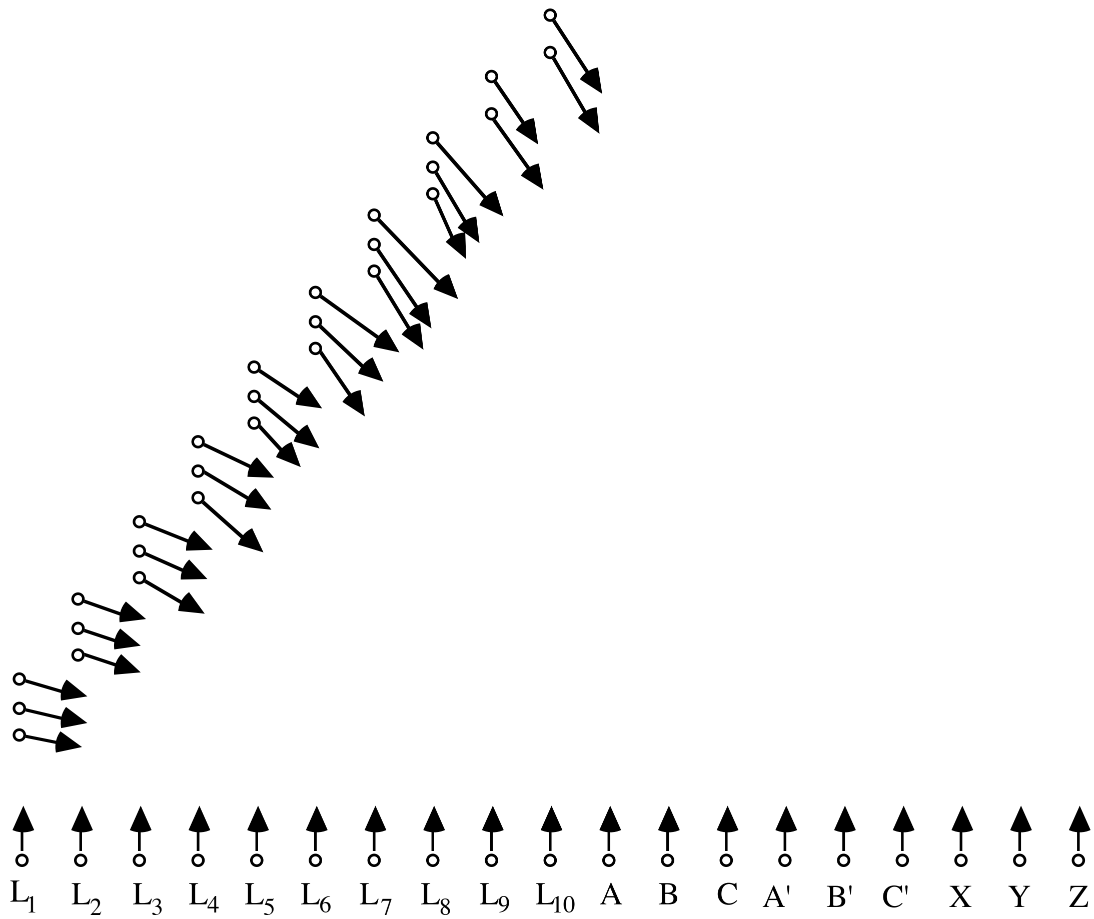


Figure 6

We turn now to the proof of part II of Theorem 1. Let P be a finite ordered set, say P has n elements, and let $L=\{a_1, a_2,\ldots,a_n\}$ be a linear extension of P. Construct an n n grid and represent the elements of P along the horizontal y=0 with the points A_1, $A_2,\ldots,A_n$ arranged at unit intervals. Assign to each A_i the upward direction. As it stands, this collection of points A_i, each with upward direction, produces an antichain. Next, let $1\leq i\leq n$ be arbitrary and let $a_{i,1}, a_{i,2}, \ldots, a_{i,k}$ be the upper covers of a_i. Consider the vertical segment from (i,i) to (i,i+1) on the plane and place points $S_{i,1}, S_{i,2},\ldots, S_{i,k}$ at $^1/_k$ intervals along this segment. Assign to each $S_{i,j}$ the direction along the line from $S_{i,j}$ pointing toward $A_{i,j}$. It is easy to verify that the point blocking relation so constructed consists precisely of P with an additional element along each covering edge, that is, a subdivision of P (cf. Figure 7).



A point blocking representation of a sub-division of Q (cf. Figure 6).

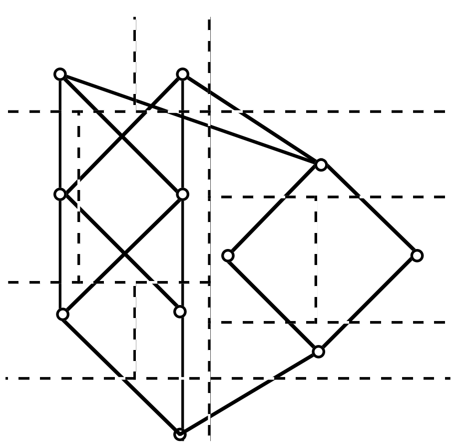
Figure 7

We shall now prove Theorem 2. Let P be a finite ordered set with a point blocking representation. We proceed by induction on the cardinality on P to show that P^{top} also has a point blocking representation, which contains the point blocking representation of P, except possibly for the directions assigned to the points corresponding to the maximal elements of P. Let $A_1, A_2, \ldots, A_k$ stand for the points representing all maximal elements of P. For each $1 \le i \le k$ construct all rays from A_i to each other point and let □ stand for the least angle between pairs of rays pointing in distinct directions. By the induction hypothesis $(P-\{a_k\})^{top}$ has a point blocking representation extending the point blocking representation for $P-\{a_k\}$ (as a subset of P) chosen above. Let T stand for the point representing the top element of $(P-\{a_k\})^{top}$. If the ray from A_k passing through T meets no other point of the representation, then we may position a top point for P as in this point blocking representation of $P - \{a_k\}$ and direct A_k toward it. Otherwise, we shift T slightly to produce a point T' which remains within an angle □ betweeen any A_i and the original T, where □=min{□$_i$ | i=1, 2,…,k. Then direct each A_i to T'. This is a point blocking representation of P^{top}.

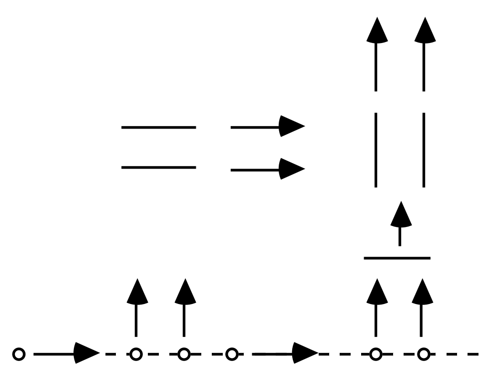
To complete the proof of Theorem 2 we show that, for the ordered set P", constructed in the proof of Theorem 1 (cf. Figure 4), P''_{bottom} has no point blocking representation. Suppose, on the contrary, that P''_{bottom} does have a point blocking representation with bottom point O. Then the points L_1, L_2,…,L_8, L" must be collinear and the direction assigned to O is along this line. Suppose the serial order of the points X, Y, Z is as illustrated in Figure 5. Then the ray from O to L" must intersect one of the line segments ZC or YC (if O lies above the XY line) or, ZC' or YC' (if O lies below the XY line). Then the direction assigned to O will not pass through L_8 or L_7, or else, L_6 or L_4. The other cases with serial order XZY, YZX, etc. are similar. This completes the proof of Theorem 2.

Line Blocking Relations

Call a finite ordered set P *series-parallel* if it can be constructed from singletons using only the operations of disjoint sum (parallel) P_1+P_2 (x and y noncomparable for each $x \in P_1$ and for each $y \in P_2$) and linear sum (series) $P_1 * P_2$ ($x < y$ for each $x \in P_1$ and for each $y \in P_2$).



Series-parallel



A line-blocking representation

Figure 8

We shall prove Theorem 3 by induction on the cardinality. In particular, our inductive hypothesis is that every series-parallel ordered set P has a line blocking relation satisfying these two properties.

(i) The directions of the line segments (possibly points) representing the maximal elements are all parallel and point in the same direction, vertical upward.

(ii) Every minimal element of P, which is not at the same time a maximal element, is represented by a point, whose assigned direction is not parallel to the vertical, and the direction opposite to its direction of motion does not meet any other line segment in the representation of P.

If P consists of a single element then such a representation is easy to construct. Let P_1, P_2 be series-parallel ordered sets with line blocking representations satisfying these two properties. We shall show that both P_1+P_2 and P_1*P_2 do too. For the purposes of the construction it is convenient to assign to a line blocking representation a convex region, on the plane, containing all line segments as well as all intersections of the directions assigned to these line segments. Let C(P) stand for such a convex region for P. (We may visualize it as a disk on the plane, say.)

First we consider the case $P = P_1+P_2$. Take the infinite region R on the plane, outside $C(P_1)$, between the rightmost direction vector assigned to a maximal of P_1 and the first ray met in the clockwise orientation, either along a direction vector, or along an opposite direction vector, or along a (non-point) line segment. Now place $C(P_2)$ anywhere in R so that the direction of its maximals are vertical upward, just like the maximals of P_1. Then a translation of $C(P_2)$ in R will suffice to ensure that no direction vector or opposite direction vector of P_2 will hit $C(P_1)$ at all. By construction no direction vector of P_1 hits $C(P_2)$. The opposite direction vectors of the minimals of P_1 and P_2 may now intersect only outside of $C(P_1)$□$C(P_2)$. This then is a line blocking representation of P_1+P_2.

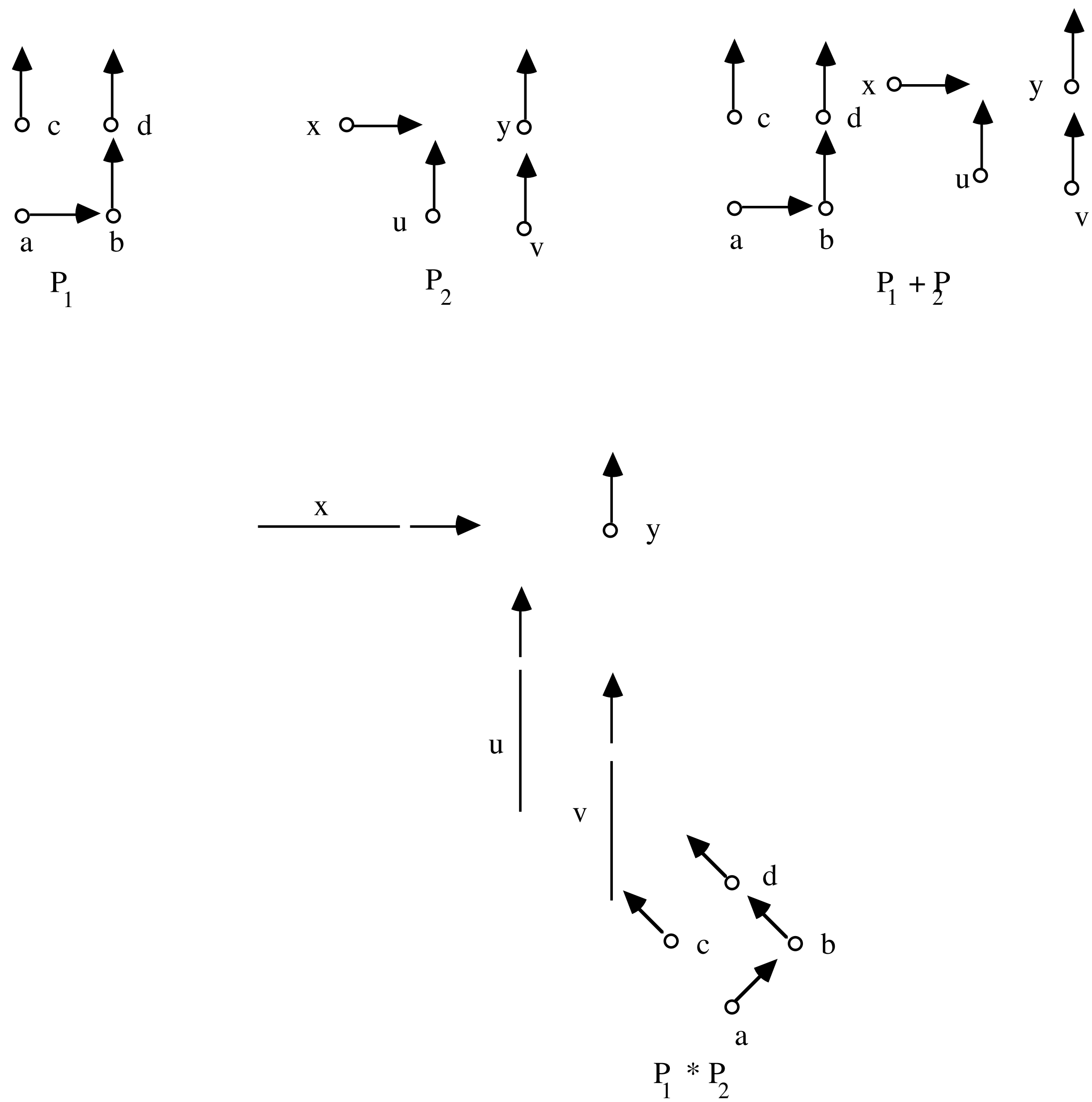


Figure 9

To construct a line blocking representation of $P_1 * P_2$ we begin again with the infinite region R constructed for P_1. First put $C(P_2)$ in R, just as above, so that the directions of its maximals are vertical upward. Translate $C(P_2)$ in R to ensure that no direction vector, or opposite direction vector, of P_2 hits $C(P_1)$ and, such that each direction vector, or opposite direction vector, of P_2 hits all of the extended vertical upward direction vectors of the maximals of the blocking relation of P_1. Moreover, by taking $C(P_2)$ far enough out in R we may suppose that these direction vectors, or opposite direction vectors, of P_2 do not intersect each other among the extended vertical upward direction vectors of the maximals of P_1. Now replace each point in

$C(P_2)$ representing a minimal element of P_2 by a line segment passing through the extended vertical upward direction vectors of the maximals of P_1. ($C(P_2)$ may be placed so far away in R that no direction vector or opposite direction vector or a minimal of P_1 hits any of these new line segments, too.) To complete the construction we must assign directions to the new line segments. For each we choose a direction along the line segment. For any minimal of P_2 whose opposite direction vector hits its corresponding line segment we use the same direction vector for the new line segment replacing it. According to the inductive hypothesis on P_2 such a direction vector for a new line segment hits only those line segments of P_2 which are bigger in the order of P_2. For any minimal of P_2 whose direction vector hits its corresponding line segment we assign the opposite direction vector. Again, the direction of such a new line segment hits only those line segments of P_2 bigger in the order. This completes the construction of $P_1 * P_2$, and therefore the proof of the theorem.

Two Directions

A finite ordered set P is an *interval order* if its elements can be represented by closed intervals along the horizontal x-axis and ordered by $I_1 \leq I_2$ if the right endpoint of I_1 is less than or equal to the left endpoint of I_2. There are, in fact, many equivalent descriptions of interval orders. Another useful one is that for any four-tuple of elements $a<b$, $c<d$ in P then, either $a \leq d$ or $c \leq b$. Thus an interval order cannot contain, up to (order) isomorphism, the disjoint sum of two chains each with at least two elements. Here is one useful consequence. Let $M_1 = \min P$ stand for the minimal elements of P, $M_2 = \min (P - M_1)$, $M_3 = \min(P - (M_1 \cup M_2)$, etc. It is not hard to verify that the elements a^i_1, a^i_2, a^i_3 of M_i can be so labelled that the upper covers of a^i_1 in M_{i+1} contain the upper covers of a^i_2 in M_{i+1}, whose upper covers in M_{i+1}, in turn, contain the upper covers of a^i_3 in M_{i+1}, etc. To prove Theorem 4, assign to each minimal element of P (that is, each element of M_1) a vertical unit length line segment and enumerate these line segments from left to right according to increasing subscript and assign to each the vertical upward direction. Next, the elements a^2_1, a^2_2, a^2_3 are represented by horizontal line segments enumerated from bottom to top, according to increasing subscript, and with lengths such that A^2_j lies above the line segment A^1_i just if $a^1_i < a^2_j$. Each of these horizontal line segments is assigned the horizontal direction to the right. The elements a^3_1, a^3_2, a^3_3,… are then represented by vertical line segments enumerated from left to right again, according to increasing subscript, with lengths so chosen that A^3_j lies to the right of A^2_i if $a^2_i < a^3_j$. Again, these vertical line segments are all assigned the same upward direction as the line segments for M_1. We may continue the construction in this way alternating horizontal and vertical line segments. This completes the proof.

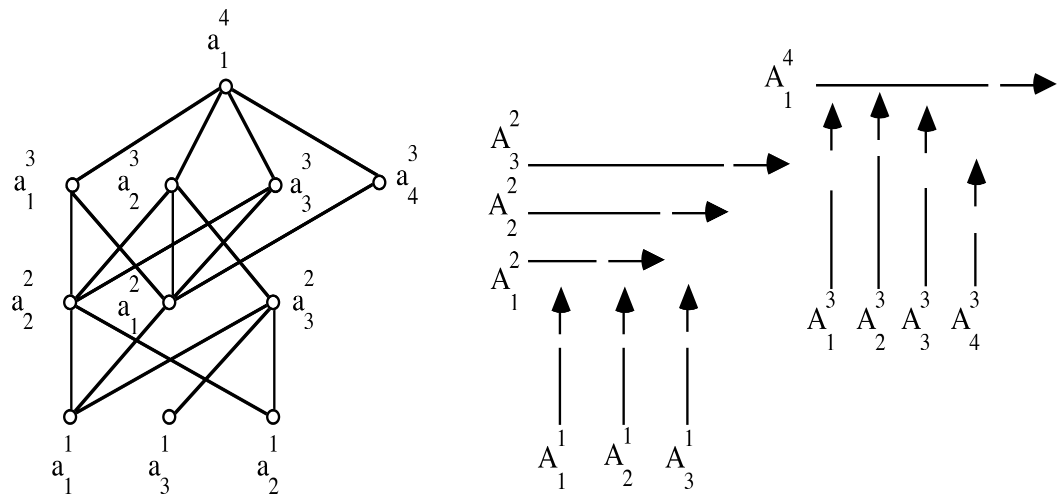


Figure 10

References

L. J. Guibas and F. F. Yao (1980) On translating a set of rectangles. *Proc. 12th Annual ACM Symposium Th. of Comp.*, 154-160.

I. Rival and J. Urrutia Representing orders on the plane by translating convex figures. *Order* **4**, (1998), 319-339.