On k-Gons and k-Holes in Point Sets*

Oswin Aichholzer† Ruy Fabila-Monroy‡ Hernán González-Aguilar §
Thomas Hackl† Marco A. Heredia ¶ Clemens Huemer‖ Jorge Urrutia**
Pavel Valtr†† Birgit Vogtenhuber†

August 29, 2014

Abstract

We consider a variation of the classical Erdős-Szekeres problems on the existence and number of convex k-gons and k-holes (empty k-gons) in a set of n points in the plane. Allowing the k-gons to be non-convex, we show bounds and structural results on maximizing and minimizing their numbers. Most noteworthy, for any k and sufficiently large n, we give a quadratic lower bound for the number of k-holes, and show that this number is maximized by sets in convex position.

1 Introduction

Let S be a set of n points in general position in the plane (i.e, no three points of S are collinear). A k-gon is a simple polygon spanned by k points of S. A k-hole is an empty k-gon; that is, a k-gon that contains no points of S in its interior.

Around 1933 Esther Klein raised the following question, which was (partially) answered in the classical paper by Erdős and Szekeres [19] in 1935: "Is it true that for any k there is a smallest integer $g(k)$ such that any set of $g(k)$ points contains at least one convex k-gon?" As observed by Klein, $g(4) = 5$, and Kalbfleisch et al. [30] proved that $g(5) = 9$. The case $k = 6$ was only solved as recently as 2006 by Szekeres and Peters [36]. They showed that $g(6) = 17$ by an exhaustive computer search. The well known Erdős–Szekeres Theorem [19] states that $g(k)$ is finite for any k. The current best bounds are $2^{k-2} + 1 \leq g(k) \leq \binom{2k-5}{k-2} + 1$ for $k \geq 5$, where the lower bound goes back to Erdős and Szekeres [20] and is conjectured to be tight. There have been many improvements on the upper bound, where the currently best bound has been obtained in 2005 by Tóth and Valtr [37]; see e.g. [3] for more details.

Erdős and Guy [18] posed the following generalization: "What is the least number of convex k-gons determined by any set S of n points in the plane?" The trivial solution for the case $k = 3$ is $\binom{n}{3}$. For convex 4-gons this question is highly non-trivial, as it is related to the search for the rectilinear crossing number $\overline{\text{cr}}(n)$, the minimum number of crossings in a straight-line drawing of the complete graph with n vertices; see the next section for details.

*Research of Oswin Aichholzer and Birgit Vogtenhuber supported by the ESF EUROCORES programme EuroGIGA – CRP 'ComPoSe', Austrian Science Fund (FWF): I648-N18. Research of Ruy Fabila-Monroy partially supported by CONACyT (Mexico), grant 153984. Research of Thomas Hackl supported by the Austrian Science Fund (FWF): P23629-N18 'Combinatorial Problems on Geometric Graphs'. Research of Clemens Huemer partially supported by projects MTM2012-30951 and Gen. Cat. DGR 2009SGR1040. Research of Marco Antonio Heredia, Hernán González-Aguilar, and Jorge Urrutia partially supported by CONACyT (Mexico), grant CB-2007/80268. Work by Pavel Valtr supported by project 1M0545 of the Ministry of Education of the Czech Republic.

†Institute for Software Technology, University of Technology, Graz, Austria, [oaich|thackl|bvogt]@ist.tugraz.at

‡Departamento de Matemáticas, Cinvestav, D.F. México, México, ruyfabila@math.cinvestav.edu.mx

§Facultad de Ciencias, Universidad Autónoma de San Luis Potosí, San Luis Potosí, México, hernan@fc.uaslp.mx

¶Departamento de Sistemas, Universidad Autónoma Metropolitana - Azcapotzalco, D.F. México, México, marco@ciencias.unam.mx

‖Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, Barcelona, Spain, clemens.huemer@upc.edu

**Instituto de Matemáticas, Universidad Nacional Autónoma de México, D.F. México, México, urrutia@matem.unam.mx

††Department of Applied Mathematics and Institute for Computer Science (ITI), Charles University, Prague, Czech Republic

In 1978 Erdős [17] raised the following question for convex k-holes: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?" As observed by Esther Klein, every set of 5 points determines a convex 4-hole, and Harborth [27] showed that 10 points always contain a convex 5-hole. Surprisingly, in 1983 Horton showed that there exist arbitrarily large sets of points containing no convex 7-hole [29]. It took almost a quarter of a century after Horton's construction to answer the existence question for $k = 6$. In 2007/08 Nicolás [33] and independently Gerken [26] proved that every sufficiently large point set contains a convex 6-hole; see also [40].

A natural generalization of the existence question for k-holes is this: "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?" Horton's construction implies $h_k(n) = 0$ for $k \geq 7$. Table 1 shows the current best lower and upper bounds for $k = 3, \ldots, 6$.

$$\begin{aligned} n^2 - \tfrac{32}{7}n + \tfrac{22}{7} &\leq h_3(n) \leq 1.6196n^2 + o(n^2) \\ \tfrac{n^2}{2} - \tfrac{9}{4}n - o(n) &\leq h_4(n) \leq 1.9397n^2 + o(n^2) \\ \tfrac{3n}{4} - o(n) &\leq h_5(n) \leq 1.0207n^2 + o(n^2) \\ \tfrac{n}{229} - 4 &\leq h_6(n) \leq 0.2006n^2 + o(n^2) \end{aligned}$$

Table 1: Bounds on the numbers $h_k(n)$ of convex k-holes [7, 12, 41].

All upper bounds in the table are due to Bárány and Valtr [12]. They are obtained by improving constructions that had been developed by Dumitrescu [15] and previously improved by Valtr [39].

Concerning the lower bounds for $k \leq 5$, Dehnhardt [14] showed in his PhD thesis that for $n \geq 13$, $h_3(n) \geq n^2 - 5n + 10$, $h_4(n) \geq \binom{n-3}{2} + 6$, and $h_5(n) \geq 3\left\lfloor \frac{n}{12} \right\rfloor$. As this PhD thesis was published in German and is not easy to access, later on several weaker bounds have been published. Only very recently these results have been subsequently improved [23, 24, 8, 9, 41, 7], where the currently best bounds can be found in [7], using a remarkable result from [24]. A result of independent interest is by Pinchasi et al. [34], who showed $h_4(n) \geq h_3(n) - \frac{n^2}{2} - O(n)$ and $h_5(n) \geq h_3(n) - n^2 - O(n)$. By this, improving the n^2-factor in the lower bound of $h_3(n)$ implies better lower bounds also for $h_4(n)$ and $h_5(n)$. Concerning lower bounds on the number of 6-holes, a proof of $h_6(n) \geq \lfloor \frac{n-1}{858} \rfloor - 2$ is contained in the proceedings version [5] of the paper at hand. This proof is based on $h_6(1717) \geq 1$ by Gerken [26]. Valtr [41] presented an improved bound of $h_6(n) \geq \frac{n}{229} - 4$, combining a different proof technique with Koshelev's result of $h_6(463) \geq 1$ [31]. As combining this result by Koshelev with the proof of [5] only gives $h_6(n) \geq \frac{n}{231} - O(1)$, the according proof is omitted here.

In this paper we generalize the above questions on the numbers of k-gons and k-holes by allowing the gons/holes to be non-convex. Thus, whenever we refer to a (general) k-gon or k-hole, unless it is specifically stated to be convex or non-convex, it could be either. Similar results for 4-holes and 5-holes can be found in [6] and [9], respectively. The PhD thesis [42] summarizes most results obtained for $k \geq 4$. See also [3] for a survey on the history of questions and results about k-gons and k-holes. We remark that in some related literature, k-holes are assumed to be convex.

A set of k points in convex position spans precisely one convex k-hole. In contrast, a point set might admit exponentially many different polygonizations (spanning cycles) [25]. Thus, the number of k-gons and k-holes can be larger than $\binom{n}{k}$, which makes the questions considered in this paper more challenging (and interesting) than they might appear at first glance.

Tables 2 and 3 summarize the best current bounds on the numbers of k-gons and k-holes, including the results of this paper. The entries in the tables list lower and upper bounds, also in explicit form if available, thus indicating for which values there are still gaps to close. Among other results, we generalize properties concerning 4-holes [6] and 5-holes [9] to $k \geq 6$. In Section 2.1 we give asymptotic bounds on the number of non-convex and general k-gons. In Section 3 we consider (general) k-holes. We show that for sufficiently small k with respect to n their number is maximized by sets in convex position, which is not the case for large k. Section 4 provides a tight bound for the maximum number of non-convex k-holes, and Section 5 contains bounds for the minimum number of general k-holes. We conclude with open problems in Section 6.

	convex	non-convex	general	
	min	max	min	max
$k=4$	$\overline{\text{cr}}(n)$ $\Theta(n^4)$	$3\binom{n}{4}-3\,\overline{\text{cr}}(n)$ $\Theta(n^4)$ [9]	$\binom{n}{4}$ $\Theta(n^4)$ [9]	$3\binom{n}{4}-2\,\overline{\text{cr}}(n)$ $\Theta(n^4)$ [9]
$k=5$	$\Theta(n^5)$ [13]	$10\binom{n}{5}-2(n-4)\,\overline{\text{cr}}(n)$ $\Theta(n^5)$ [9]	$\binom{n}{5}$ $\Theta(n^5)$ [9]	$\Theta(n^5)$ [Sec. 2.1]
$k\geq 6$	$\Theta(n^k)$ [13]	$\Theta(n^k)$ [Sec. 2.1]	$\binom{n}{k}$ $\Theta(n^k)$ [Sec. 2.1]	$\Theta(n^k)$ [Sec. 2.1]

Table 2: Bounds on the numbers of convex, non-convex, and general k-gons for n points and constant k.

	convex	non-convex	general	
	min	max	min	max
$k=4$	$\geq \frac{n^2}{2}-\frac{9}{4}n-o(n)$ $\leq 1.9397n^2+o(n^2)$ $\Theta(n^2)$ [7, 12]	$\leq \frac{n^3}{2}-O(n^2)$ $\geq \frac{n^3}{2}-O(n^2\log n)$ $\Theta(n^3)$ [6]	$\geq \frac{5}{2}n^2-O(n)$ $\leq O(n^{\frac{5}{2}}\log n)$ $\Omega(n^2)$ [6], $O(n^{\frac{5}{2}}\log n)$ [Sec. 5]	$\binom{n}{4}$ $\Theta(n^4)$ [6]
$k=5$	$\geq \frac{3n}{4}-o(n)$ $\leq 1.0207n^2+o(n^2)$ $\Omega(n)$ [7], $O(n^2)$ [12]	$\leq n!/(n-4)!$ $\Theta(n^4)$ [Sec. 4]	$\geq 17n^2-O(n)$ $\leq O(n^3(\log n)^2)$ $\Omega(n^2)$ [9], $O(n^3(\log n)^2)$ [Sec. 5]	$\binom{n}{5}$ $\Theta(n^5)$ [9]
$k\geq 6$	$k=6$: $\geq \frac{n}{229}-4$ $\Omega(n)$ [41] $O(n^2)$ [12] $k\geq 7$: $\emptyset$ [29]	$\leq n!/(n-k+1)!$ $\Theta(n^{k-1})$ [Sec. 4]	$\geq n^2-O(n)$ $\leq O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ $\Omega(n^2)$, $O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ [Sec. 5]	$\binom{n}{k}$ $\Theta(n^k)$ [Sec. 3]

Table 3: Bounds on the numbers of convex, non-convex and general k-holes for n points and constant k.

2 General k-gons

2.1 k-gons and the rectilinear crossing number

For small values of k, the number of k-gons in a point set S of n points can be related to the rectilinear crossing number $\overline{\text{cr}}(S)$ of S. This is the number of proper intersections (i.e., intersections in the interior of edges) in the (drawing of the) complete straight-line graph on S. By $\overline{\text{cr}}(n)$ we denote the minimum possible rectilinear crossing number over all point sets of cardinality n. Determining $\overline{\text{cr}}(n)$ is a well-known problem in discrete geometry; see [13, 18] as general references and [4] for bounds on small sets. Asymptotically we have $\overline{\text{cr}}(n) = c_4\binom{n}{4} = \Theta(n^4)$, where c_4 is a constant in the range $0.379972 \leq c_4 \leq 0.380473$. The currently best lower bound on c_4 is by Ábrego et al. [2, 1]. The upper bound stems from a recent work of Fabila-Monroy and López [22].

It is easy to see that the number of convex 4-gons is equal to $\overline{\text{cr}}(S)$ and is thus minimized by sets realizing $\overline{\text{cr}}(n)$. Since four points in non-convex position span three non-convex 4-gons, we have at most $3\binom{n}{4}-3\,\overline{\text{cr}}(n) \approx 1.86\binom{n}{4}$ non-convex and at most $3\binom{n}{4}-2\,\overline{\text{cr}}(n) \approx 2.24\binom{n}{4}$ general 4-gons. These bounds are tight for point sets minimizing the rectilinear crossing number.

A similar relation has been obtained for the number of non-convex 5-gons in [9]: Any set of n points has at most $10\binom{n}{5} - 2(n-4)\,\overline{\text{cr}}(n) \approx 6.2\binom{n}{5}$ non-convex 5-gons. Again, this bound is achieved by sets minimizing the rectilinear crossing number. Note that the maximum numbers of non-convex 4- and 5-gons exceed the maximum numbers of their convex counterparts. For the number of general 5-gons, no such direct relation to $\overline{\text{cr}}(n)$ is possible, as already for $n = 6$ there exist point sets with the same number of crossings but different numbers of 5-gons [9]. Similarly, for $k \geq 6$, none of the three types of k-gons (convex, non-convex, and general) in a point set S can be expressed as a function of $\overline{\text{cr}}(S)$. Still, we can use the rectilinear crossing number to obtain bounds on these numbers. Let $g_k^t(S)$ be the number of k-gons of *type* t (*conv*ex, *non-conv*ex, or *gen*eral) in a point set S.

Proposition 1. *Let $k \geq 4$, and let c_1, c_2, and x be arbitrary fixed constants such that Inequality (1) holds for all sets S' with cardinality $|S'| = k$.*

$$c_1 \leq g_k^t(S') + x \cdot \overline{\text{cr}}(S') \leq c_2 \tag{1}$$

Then for every point set S with $|S| = n \geq k$, the following bounds hold for the number $g_k^t(S)$ of k-gons of type t in S.

$$g_k^t(S) \quad \geq \quad c_1 \cdot \binom{n}{k} - x \cdot \binom{n-4}{k-4} \cdot \overline{\text{cr}}(S) \tag{2}$$

$$g_k^t(S) \quad \leq \quad c_2 \cdot \binom{n}{k} - x \cdot \binom{n-4}{k-4} \cdot \overline{\text{cr}}(S) \tag{3}$$

Proof. Given some point set S with n points, consider all its $\binom{n}{k}$ subsets of size k $\{S_i \subseteq S : |S_i| = k\}$. Then we have the following equations.

$$\sum_i \overline{\text{cr}}(S_i) \quad = \quad \binom{n-4}{k-4} \cdot \overline{\text{cr}}(S) \tag{4}$$

$$\sum_i g_k^t(S_i) \quad = \quad g_k^t(S) \tag{5}$$

Using the first inequality in (1), Equation (5) can be transformed to the lower bound

$$g_k^t(S) \ = \ \sum_i g_k^t(S_i) \ \geq \ \sum_i (c_1 - x \cdot \overline{\text{cr}}(S_i)) \ = \ c_1 \cdot \binom{n}{k} - x \cdot \sum_i \overline{\text{cr}}(S_i)$$

which, by applying (4), gives the desired bound (2). Analogously, we obtain (3) if we combine the second inequality in (1) with Equations (4) and (5). □

If x is negative in Inequality (2), then the rectilinear crossing number $\overline{\text{cr}}(S)$ of S adds to the lower bound. Thus we can replace it by the minimum over all point sets of size n, $\overline{\text{cr}}(n)$, and by this obtain a lower bound that is independent of S.

Corollary 2. *Assume that for some constants $x \leq 0$ and c_1 arbitrary, the inequality $c_1 \leq g_k^t(S') + x \cdot \overline{\text{cr}}(S')$ is satisfied for all point sets S' with cardinality $|S'| = k$. Then for every point set S with $|S| = n \geq k$ the following lower bound holds for the number $g_k^t(S)$ of k-gons of type t in S.*

$$g_k^t(S) \geq c_1 \cdot \binom{n}{k} - x \cdot \binom{n-4}{k-4} \cdot c_4 \binom{n}{4} = \left(c_1 - x \cdot c_4 \cdot \binom{k}{4}\right) \binom{n}{k} \tag{6}$$

Accordingly, if x is positive, we can generalize Inequality (3) to a general upper bound.

Corollary 3. *Assume that for some constants $x \geq 0$ and c_2 arbitrary, the inequality $c_2 \geq g_k^t(S') + x \cdot \overline{\text{cr}}(S')$ is satisfied for all point sets S' with cardinality $|S'| = k$. Then for every point set S with $|S| = n \geq k$, the following upper bound applies to the number $g_k^t(S)$ of k-gons of type t in S.*

$$g_k^t(S) \leq \left(c_2 - x \cdot c_4 \cdot \binom{k}{4}\right) \binom{n}{k} \tag{7}$$

In each of the bounds resulting from Proposition 1, either c_1 or c_2 is not used. So of course, for independent optimization of the two bounds, it might be helpful to consider the pairs (c_1, x), and (c_2, x) independently, with possibly different optimal values for x. On the other hand, optimizing all three values c_1, c_2, and x simultaneously results in bounds that are more easy to compare. In the following we optimize in these two different ways. On the one hand we try to minimize the difference between c_1 and c_2 to obtain most possibly small ranges for the number of (some type of) k-gons in sets with a certain rectilinear crossing number. On the other hand we independently optimize (c_1, x) and (c_2, x) in order to obtain general bounds (meaning bounds that are independent from the rectilinear crossing number of a given set) by applying Corollaries 2 and 3 .

Note that concerning the general bounds, lower bounds only make sense for the classical question about convex k-gons, as the numbers of general and non-convex k-gons are minimized by sets in convex position. Similarly, general upper bounds for non-convex or (general) k-gons are of interest while the maximum number of convex k-gons is $\binom{n}{k}$, again achieved by convex sets.

Recall that the number of k-gons (of whatever type) in a point set S only depends on the combinatorial properties and thus on the order type $OT(S)$ of the underlying point set S. Thus, we calculate pairs $(g_k^t(OT), \overline{\text{cr}}(OT))$ for all possible order types OT of k points, this way obtaining all possible pairs $(g_k^t(S'), \overline{\text{cr}}(S'))$ that can occur for any point set S' with $|S'| = k$. For the calculation, we use the order type database [10], that contains a complete list of the order types of up to 11 points. Having this, we can optimize the values c_1, c_2, and x that fulfill (1), obtaining the according relations (2) and (3). Tables 4 and 5 show an overview of the resulting relations. Note that trivial bounds on the numbers of k-gons can be obtained by assuming the theoretic possible maximum number of k-gons for each k-tuple. As the maximum numbers of general / non-convex k-gons in a k-tuple are 8, 29, and 92 for $k \in \{5, 6, 7\}$, the bounds in Table 5 all improve over these trivial bounds.

$$
\begin{array}{rcl}
-0.75\binom{n}{5} + 0.25 \cdot (n-4) \cdot \overline{\text{cr}}(S) \;\leq & g_5^{conv}(S) & \leq\; -0.25\binom{n}{5} + 0.25 \cdot (n-4) \cdot \overline{\text{cr}}(S) \\
& g_5^{non-conv}(S) & =\; 10\binom{n}{5} - 2 \cdot (n-4) \cdot \overline{\text{cr}}(S) \\
9.25\binom{n}{5} - 1.75 \cdot (n-4) \cdot \overline{\text{cr}}(S) \;\leq & g_5^{gen}(S) & \leq\; 9.75\binom{n}{5} - 1.75 \cdot (n-4) \cdot \overline{\text{cr}}(S) \\
-\binom{n}{6} + 0.08\dot{3}\binom{n-4}{2} \cdot \overline{\text{cr}}(S) \;\leq & g_6^{conv}(S) & \leq\; -0.25\binom{n}{6} + 0.08\dot{3}\binom{n-4}{2} \cdot \overline{\text{cr}}(S) \\
29\tfrac{4}{9}\binom{n}{6} - \tfrac{22}{9}\binom{n-4}{2} \cdot \overline{\text{cr}}(S) \;\leq & g_6^{non-conv}(S) & \leq\; 36\tfrac{6}{9}\binom{n}{6} - \tfrac{22}{9}\binom{n-4}{2} \cdot \overline{\text{cr}}(S) \\
28\tfrac{1}{3}\binom{n}{6} - \tfrac{7}{3}\binom{n-4}{2} \cdot \overline{\text{cr}}(S) \;\leq & g_6^{gen}(S) & \leq\; 36\binom{n}{6} - \tfrac{7}{3}\binom{n-4}{2} \cdot \overline{\text{cr}}(S) \\
-1.1\overline{923076}\binom{n}{7} + 0.0\overline{384615}\binom{n-4}{3} \cdot \overline{\text{cr}}(S) \;\leq & g_7^{conv}(S) & \leq\; -0.3\overline{461538}\binom{n}{7} + 0.0\overline{384615}\binom{n-4}{3} \cdot \overline{\text{cr}}(S) \\
86.\overline{230769}\binom{n}{7} - 3.\overline{538461}\binom{n-4}{3} \cdot \overline{\text{cr}}(S) \;\leq & g_7^{non-conv}(S) & \leq\; 123.\overline{846153}\binom{n}{7} - 3.\overline{538461}\binom{n-4}{3} \cdot \overline{\text{cr}}(S) \\
85.5\binom{n}{7} - 3.5\binom{n-4}{3} \cdot \overline{\text{cr}}(S) \;\leq & g_7^{gen}(S) & \leq\; 123.5\binom{n}{7} - 3.5\binom{n-4}{3} \cdot \overline{\text{cr}}(S)
\end{array}
$$

Table 4: Bounding the number of k-gons in an n point set S via its rectilinear crossing number $\overline{\text{cr}}(S)$.

$g_5^{non-conv}(S)$	$\leq$	$(10 - 10 \cdot c_4)\binom{n}{5}$	$\approx$	$6.20\binom{n}{5}$
$g_5^{gen}(S)$	$\leq$	$(9.75 - 8.75 \cdot c_4)\binom{n}{5}$	$\approx$	$6.43\binom{n}{5}$
$g_6^{non-conv}(S)$, $g_6^{gen}(S)$	$\leq$	$(36 - 35 \cdot c_4)\binom{n}{6}$	$\approx$	$22.7\binom{n}{6}$
$g_7^{non-conv}(S)$	$\leq$	$(123.\overline{846153} - 123.\overline{846153} \cdot c_4)\binom{n}{7}$	$\approx$	$75.64\binom{n}{7}$
$g_7^{gen}(S)$	$\leq$	$(123.5 - 122.5 \cdot c_4)\binom{n}{7}$	$\approx$	$76.95\binom{n}{7}$

Table 5: Bounding the number of k-gons in an n point set S via the constant c_4 of the (minimum) rectilinear crossing number $\overline{\text{cr}}(n)$, $\overline{\text{cr}}(n) = c_4\binom{n}{4}$.

From the calculations it can be seen that for all sets of size $k \leq 7$, the point sets reaching the maximum number of general or non-convex k-gons are at the same time minimizing the number of crossings. The same is true for $k = 8$. But continuing the calculations until $k = 9$, it turns out that this is not true in general. The (combinatorially unique) point set containing the maximum number of 1282 general 9-gons has 38 crossings and thus does not reach the (minimum) rectilinear crossing number $\overline{\text{cr}}(9) = 36$ [4].

2.2 k-gons, polygonizations, and the double chain

Polygonizations, also called spanning cycles, can be considered as k-gons of maximal size (i.e., $k = n$). García et al. [25] construct a point set with $\Omega(4.64^n)$ spanning cycles, the so-called double chain $DC(n)$, which is currently the best known minimizing example; see Figure 1.

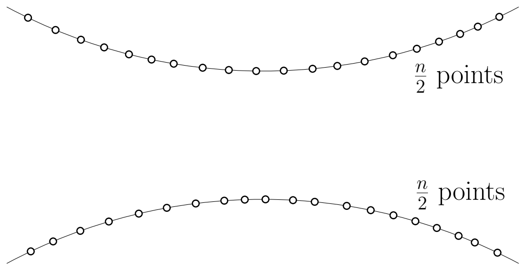

Figure 1: The so-called double chain $DC(n)$.

The upper bound on the number of spanning cycles of any n-point set was improved several times during the last years, most recently to $O(68.664^n)$ [16] and $O(54.543^n)$ [35], neglecting polynomial factors in the asymptotic expressions. The minimum is achieved by point sets in convex position, which have exactly one spanning cycle. For the number of general k-gons this implies a lower bound of $\binom{n}{k}$, as well as an upper bound of $O\left((54.543^k\binom{n}{k}\right)$. Hence, for constant k, any point set has $\Theta(n^k)$ general k-gons. On the other hand, the double chain provides $\Omega(n^k)$ non-convex k-gons, where $k \geq 4$ is again a constant. To see this, choose one vertex from the upper chain of $DC(n)$ and $k-1 \geq 3$ vertices from the lower chain of $DC(n)$, and connect them to a simple, non-convex polygon. This gives at least $\frac{n}{2}\binom{n/2}{k-1} = \Omega(n^k)$ non-convex k-gons. As the lower bound on the maximal number of non-convex k-gons asymptotically matches the upper bound on the maximal number of general k-gons, we obtain the following result.

Proposition 4. *For any constant $k \geq 4$, the number of non-convex k-gons in a set of n points is bounded by $O(n^k)$. This is tight in the sense that there exist sets with $\Omega(n^k)$ non-convex k-gons.*

3 Maximizing the number of (general) k-holes

In [6] it is shown that the number of 4-holes is maximized for point sets in convex position if n is sufficiently large. It was conjectured that this is true for any constant $k \geq 4$. The following theorem settles this conjecture in the affirmative.

Theorem 5. *For every $k \geq 4$ and $n \geq 2(k-1)!\binom{k}{4}+k-1$, the number of k-holes is maximized by a set of n points in convex position.*

Proof. Consider a non-convex k-hole H. For each of its non-extreme vertices (i.e., vertices not on the convex hull of H), there exists a triangle spanned by three extreme vertices of H such that the non-extreme vertex is contained in the interior of the triangle. Further, there exists at least one reflex (and thus non-extreme) vertex v_r of H such that removing v_r from the vertex set of H (and connecting its incident vertices) results in a simple non-empty $(k-1)$-gon. To see the latter, consider an edge e of $\mathrm{CH}(H)$ which is not in the boundary of H. Together with some part of the boundary of H, e forms a simple polygon H' that is interior-disjoint with H. For the case where H' is just a triangle, e can be used to cut off the third vertex of H' from H. Clearly, the resulting $(k-1)$-gon is simple. If H' has at least four vertices then any triangulation of H' has at least two ears. For any ear not incident to e, the according diagonal of the triangulation can be used to cut off the central vertex of the ear from H. Again, the resulting $(k-1)$-gon is simple.

Now consider a non-empty triangle Δ. We give an upper bound for the number of non-convex k-holes having the three vertices of Δ as extreme points. Denote by $\mathcal{K}$ the set of simple non-empty $(k-1)$-gons having the vertices of Δ on their convex hull. First, $|\mathcal{K}|$ can be bounded from above by the number of simple, possibly empty $(k-1)$-gons having the three vertices of Δ on their boundary, that is, $|\mathcal{K}| \leq \frac{(k-2)!}{2}\binom{n-3}{k-4}$.

Further, every simple $(k-1)$-gon in $\mathcal{K}$ may be completed to a simple non-convex k-hole in at most $k-1$ ways by adding a reflex vertex: As the resulting polygon has to be empty, we have to use the inner geodesic connecting the two adjacent vertices of the $(k-1)$-gon. Only if this geodesic contains exactly one point, we do obtain one non-convex k-hole. Thus the number of non-convex k-holes having all vertices of Δ on their convex hull is bounded from above by

$$(k-1)\frac{(k-2)!}{2}\binom{n-3}{k-4} = \frac{(k-1)!}{2}\binom{n-3}{k-4}.$$

Considering convex k-holes, observe that every k-tuple gives at most one convex k-hole. Denote by N the number of k-tuples that do *not* form a convex k-hole, and by T the number of non-empty triangles. Then we get (8) as a first upper bound on the number of (general) k-holes of a point set.

$$\binom{n}{k} - N + \left(\frac{(k-1)!}{2}\binom{n-3}{k-4}\right) \cdot T \tag{8}$$

To obtain an improved upper bound from (8), we need to derive a good lower bound for N. To this end, consider again a non-empty triangle Δ. As Δ is not empty, none of the $\binom{n-3}{k-3}$ k-tuples that contain all three vertices of Δ forms a convex k-hole. On the other hand, for such a k-tuple, all of its $\binom{k}{3}$ contained triangles might be non-empty. Thus, we obtain $T \cdot \binom{n-3}{k-3}/\binom{k}{3}$ as a lower bound for N and (9) as an upper bound for the number of k-holes.

$$\binom{n}{k} + \left(\frac{(k-1)!}{2}\binom{n-3}{k-4} - \frac{\binom{n-3}{k-3}}{\binom{k}{3}}\right) \cdot T \tag{9}$$

For $n \geq 2(k-1)!\binom{k}{4} + k - 1$ this is at most $\binom{n}{k}$, the number of k-holes of a set of n points in convex position, which proves the theorem. □

The above theorem states that convexity maximizes the number of k-holes for $k = O(\frac{\log n}{\log\log n})$ and sufficiently large n. Moreover, the proof implies that any non-empty triangle in fact reduces the number of empty k-holes. Thus it follows that, for $k = O(\frac{\log n}{\log\log n})$ and n sufficiently large, the maximum number of convex k-holes is strictly larger than the maximum number of non-convex k-holes; see also the next section.

At the other extreme, for $k \approx n$ the statement does not hold: As already mentioned in the introduction, a set of k points spans at most one convex k-gon, but might admit exponentially many different non-convex k-gons [25]. This leads to the question, for which k the situation changes. The following theorem implies that for some $0 < c < 1$ and every $k \geq c \cdot n$, the convex set does not maximize the number of k-holes.

Theorem 6. *The number of k-holes in the double chain $DC(n)$ on n points is at least*

$$\binom{\frac{n-4}{2}}{\frac{n-k}{2}} \cdot \frac{n-k+2}{2} \cdot \Omega(4.64^k).$$

Proof. Recall that García et al. [25] showed that the double chain on n points ($n/2$ points on each chain) admits $\Omega(4.64^n)$ polygonizations. To estimate the number of k-holes of the double chain on n points, we first use this result for a double chain on k points ($k/2$ points on each chain), obtaining $\Omega(4.64^k)$ different k-polygonizations. Then we distribute the remaining $n-k$ points among all possible positions, meaning that for each k-polygonization, we obtain the double chain on n points with a k-hole drawn; see Figure 3.

In their proof, García et al. count paths that start at the first vertex of the upper chain and end at the last vertex of the lower chain. Before the first vertex on the lower chain, they add an additional point q to complete these paths to polygonizations. We slightly extend this principle, by also adding an additional point p on the upper chain after the last vertex; see Figure 2.

Then we complete each path C to a polygonization in one of the following ways: Either we add p to C directly next to $p_{\frac{k}{2}-1}$ and then complete C via q, obtaining P_q, or we add q to C directly next to q_1, and close the polygonization via p, obtaining P_p; see again Figure 3.

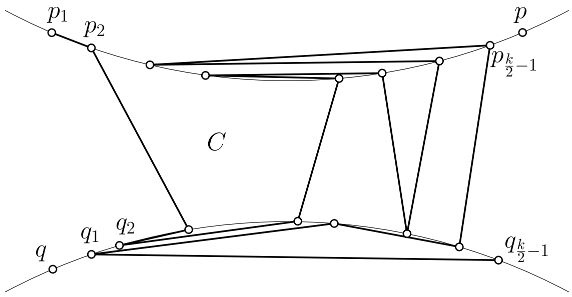

Figure 2: A path C in the double chain, using all but the vertices p and q.

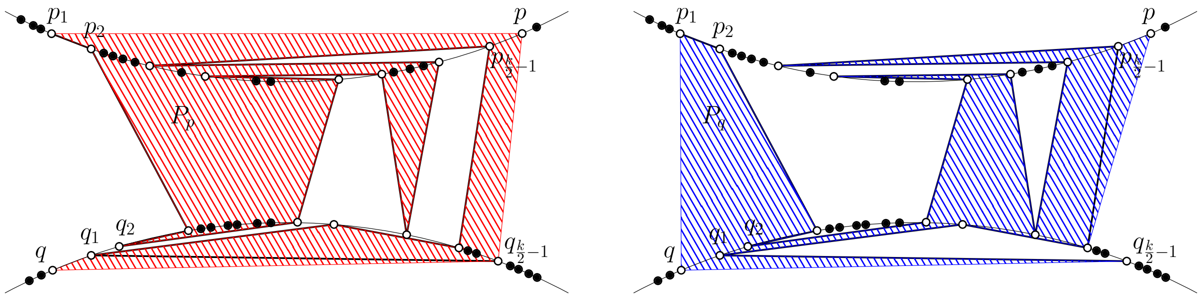

Figure 3: Two ways to complete a path to a polygonization.

Note that this changes the number of polygonizations only by a constant factor and thus does not influence the asymptotic bound. However, the interior of P_q is the exterior of its "complemented" polygonization P_p, meaning that if we place a point somewhere on the double chain and it lies inside P_q, then it lies outside P_p, and vice versa. It follows that, in one of the two polygonizations, at least half of the $k+2$ positions to insert points are outside the polygonization. Hence, we can distribute the $\frac{n-k}{2}$ points on each chain to at least $\frac{k}{2}+1$ possible positions in total. Now, on one of the two chains we have at least $\frac{k}{4}+1$ positions; see again Figure 3. More precisely, there are $\frac{k}{4}+j+1$ positions on this chain (where $0 \leq j < \frac{k}{4}$) and (at least) $\max\{2, \frac{k}{4}-j\}$ positions on the other chain. The lower bound stems from the fact that the positions before the first and after the last vertex of a chain are always possible. Placing a points on the b positions of one chain can be seen as placing a balls into b boxes. The number of ways to do so is $\binom{a+b-1}{a}$. Using this, we obtain

$$\binom{\frac{n-k}{2}+\frac{k}{4}+j}{\frac{n-k}{2}} \cdot \max\left\{\binom{\frac{n-k}{2}+1}{\frac{n-k}{2}}, \binom{\frac{n-k}{2}+\frac{k}{4}-j-1}{\frac{n-k}{2}}\right\}$$

possibilities to place the remaining points on the two chains. This factor is minimized for $j = \frac{k}{4}-2$, which yields the claimed lower bound of

$$\binom{\frac{n-4}{2}}{\frac{n-k}{2}} \cdot \frac{n-k+2}{2} \cdot \Omega(4.64^k)$$

for the number of k-holes of $DC(n)$. □

4 An upper bound for the number of non-convex k-holes

The following theorem shows that for sufficiently small k with respect to n, the maximum number of non-convex k-holes is smaller than the maximum number of convex k-holes.

Theorem 7. *For any constant $k \geq 4$, the number of non-convex k-holes in a set of n points is bounded by $O(n^{k-1})$ and there exist sets with $\Omega(n^{k-1})$ non-convex k-holes.*

Proof. We first show that there are at most $O(n^{k-1})$ non-convex k-holes by giving an algorithmic approach to generate all non-convex k-holes. We represent a non-convex k-hole by the counter-clockwise sequence of its vertices, where we require that the last vertex is reflex and its removal results in a simple $(k-1)$-gon; see again the proof of Theorem 5. Any non-convex k-hole has $r \geq 1$ such representations, where r is at most the number of its reflex vertices. Thus the number of different representations is an upper bound on the number of non-convex k-holes.

For $1 \leq i \leq k-1$, we have $n-i+1$ possibilities to choose the i-th vertex v_i. If the resulting $(k-1)$-gon is non-simple, we ignore it (but still count it). For the last vertex v_k, we have at most one possibility: As v_k is required to be reflex and the polygon has to be empty, we have to use the inner geodesic connecting v_{k-1} back to v_1. Only if this geodesic contains exactly one point, namely v_k, we do obtain one non-convex k-hole. Altogether, we obtain at most $n(n-1)(n-2)\ldots(n-k+2) = n!/(n-k+1)! = O(n^{k-1})$ non-convex k-holes.

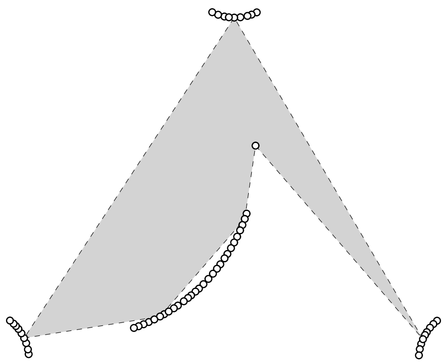

Figure 4: A set with $\Theta(n^{k-1})$ non-convex k-holes.

For an example achieving this bound see Figure 4. Each of the four indicated groups of points contains a linear fraction of the point set; for example $\frac{n}{4}$ points. To show that in this example we have $\Omega(n^{k-1})$ non-convex k-holes it is sufficient to only consider the k-holes with triangular convex hull of the type indicated in the figure. For each of the three vertices of the convex hull of the k-hole we have a linear number of possible choices, and the $k-4$ non-reflex inner vertices can also be chosen from a linear number of vertices. Hence, we obtain $\Omega\left(n^3 \cdot \binom{n}{k-4}\right) = \Omega(n^{k-1})$ non-convex k-holes. □

5 On the minimum number of (general) k-holes

We start with an upper bound on the minimum (over all n-point sets) number of (general) k-holes. Note that the minimum number of (general) k-holes cannot be greater than the minimum number of convex k-holes plus the maximum number of non-convex k-holes. Recall that the minimum number of convex k-holes is $O(n^2)$ for $k \leq 6$ (and zero for $k \geq 7$), and the maximum number of non-convex k-holes is $O(n^{k-1})$. For $k \geq 3$, the latter dominates the former, yielding an upper bound of $O(n^{k-1})$ for the minimum number of general k-holes. But this bound is by far not tight, as the following considerations show.

5.1 An upper bound on the minimum number of (general) k-holes using grids

Consider an integer grid G of size $\sqrt{n} \times \sqrt{n}$. We denote a segment spanned by two points of G that does not have any points of G in its interior as *prime segment.* Further, we denote the *slope* of a line l spanned by points of G as the differences (d_x, d_y) of the coordinates of the endpoints of a prime segment on l. Note that a line with slope $(0,1)$ or $(1,0)$ contains exactly $\sqrt{n}$ points of G. A line with slope (d_x, d_y), both $d_x, d_y \neq 0$, contains at most $\min\left\{\left\lceil \frac{\sqrt{n}}{|d_x|} \right\rceil, \left\lceil \frac{\sqrt{n}}{|d_y|} \right\rceil\right\}$ points of G. A k-gon spanned by points of G is called *interior-empty* if it does not contain any points of G in its interior.

Lemma 8. *In an integer grid G of size $\sqrt{n} \times \sqrt{n}$, every segment is incident to at most $O(\sqrt{n}\log n)$ interior-empty triangles.*

Proof. Consider an arbitrary segment pq spanned by points of G (and possibly containing some points of G in its interior), and its supporting line l. Let l' and l'' be the two lines parallel to l and going through points of G such that no point of G lies between l and l', and between l and l'', respectively; see Figure 5 where the segment pq is a prime segment.

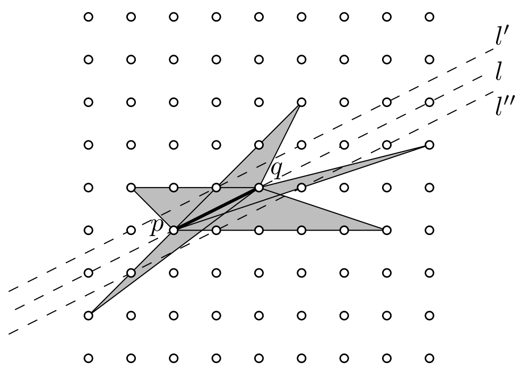

Figure 5: A prime segment pq in a 9×9 integer grid with the according lines l, l', and l''. Gray triangles are interior-empty and have the third point outside the strip $l'l''$.

Both, l' and l'', contain at most $\sqrt{n}$ points of G, each spanning an interior-empty triangle with pq. Further, each of the points on l spans a degenerate interior-empty triangle with pq. Any other triangle Δ incident to pq has its third point r strictly outside the strip bounded by l' and l''.

A necessary condition for such a triangle Δ to be interior-empty is that both, pr and qr, 'pass through' the same prime segment s on l' (or l''). Moreover, at least one of the supporting lines of pr and qr contains an endpoint of this prime segment s. To see this, w.l.o.g., let s be a prime segment on l'. Further, let l_p be the line through p and one endpoint of s and l_q be the line through q and the other endpoint of s such that l_p and l_q do not cross between l and l'. See Figure 6.

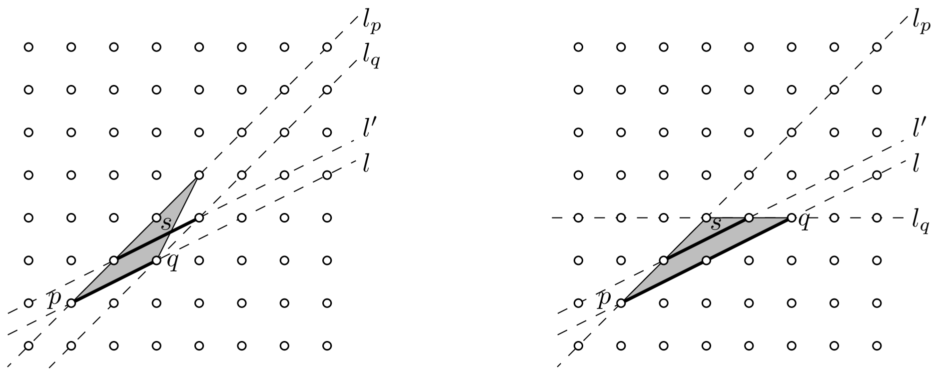

Figure 6: Lines l_p and l_q for a non-prime segment pq (left) or a prime segment pq (right).

Consider the region A that contains s and is bounded by l, l_p, and l_q. Note that A is a half-strip, if pq is a prime segment, or triangular, otherwise; see Figure 6 again. Moreover, as s is a prime segment, the interior of A does not contain any point of G. Thus, any point seen from both p and q through s lies on the boundary of A, more precisely, on l_p or l_q.

Using this latter property, we can derive an upper bound on the number of points r that are visible from p and q via the same prime segment s by counting the number of points on such supporting lines.

Consider first the case that pq is a horizontal segment, i.e., $q - p = (d_x, 0)$. Then the according lines through p (or q) and a point of a prime segment on l' (or l'') have slopes in the range $\{(0,1), (\pm 1, 1), (\pm 2, 1), \ldots, (\pm\sqrt{n}, 1)\}$. Assuming that all these lines in fact exist for pq in G, we obtain the following upper bound for the total number of interior-empty triangles incident to pq (the first term $3\sqrt{n}$ is for triangles having the third point on one of l, l', and l'').

$$3\sqrt{n} + 2 \cdot \sum_{i=-\sqrt{n}}^{-1} \left\lceil \frac{\sqrt{n}}{|i|} \right\rceil + 2 \cdot \sqrt{n} + 2 \cdot \sum_{i=1}^{\sqrt{n}} \left\lceil \frac{\sqrt{n}}{|i|} \right\rceil \quad = \quad O(\sqrt{n} \log(n)) \tag{10}$$

For the general case of pq being a segment with $q - p = (d_x, d_y)$, its supporting line l has slope $(d'_x, d'_y) = (\frac{d_x}{\gcd(d_x,d_y)}, \frac{d_y}{\gcd(d_x,d_y)})$. The according slopes of the lines through p (or q) and a point of a prime segment on l' (or l'') differ from each other by a multiple of (d'_x, d'_y). Note that d'_x and d'_y are integers with $\max\{d'_x, d'_y\} \geq 1$. Thus, the according number of interior-empty triangles for a general segment cannot exceed the bound in (10) for a horizontal segment, which completes the proof. □

Theorem 9. *For every constant $k \geq 4$ and every number $n \geq k$ with $\sqrt{n} \in \mathbb{N}$, there exist sets with n points in general position that admit at most $O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ k-holes.*

Proof. The point set S we consider is the squared Horton set of size $\sqrt{n} \times \sqrt{n}$; see [38]. Roughly speaking, S is a grid which is perturbed such that every set of originally collinear points forms a Horton set. For any two points $p, q \in S$, the number of empty triangles that have the segment pq as an edge is bound from above by the number of (possibly degenerate) interior-empty triangles incident to the according segment in the regular grid. By Lemma 8, this latter number is at most $O(\sqrt{n} \log n)$.

To estimate the number of k-holes in S, we use triangulations and their dual: In the dual graph of a triangulation, every triangle is represented as a node, and two nodes are connected iff the corresponding triangles share an edge. For the triangulation of a k-hole, this gives a binary tree which can be rooted at any triangle that has an edge on the boundary of the k-hole. It is well-known that there are $O(4^k \cdot k^{-\frac{3}{2}})$ such rooted binary trees [32]. Although exponential in k, this bound is constant with respect to the size n of S.

Now pick an empty triangle Δ in S and an arbitrary rooted binary tree B of size $k-2$. Consider all k-holes which admit a triangulation whose dual is B rooted at Δ. Note that each such k-hole contains Δ. We "build" and count (triangulations of) these k-holes by starting from Δ and following the edges of B. As by Lemma 8, the number of empty triangles incident to an edge in S is $O(\sqrt{n} \log n)$, each of the $k-3$ edges in B yields $O(\sqrt{n} \log n)$ possibilities to continue a triangulated k-hole, and we obtain an upper bound of $O((\sqrt{n} \log n)^{k-3})$ for the number of triangulations of k-holes for Δ that represent B.

Multiplying this by the (constant) number of rooted binary trees of size $k - 2$ does not change the asymptotics and thus yields an upper bound of $O((\sqrt{n} \log n)^{k-3})$ for the number of all triangulations of all k-holes containing Δ. As any k-hole can be triangulated, this is also an upper bound for the number of k-holes containing Δ.

Finally, as there are $O(n^2)$ empty triangles in S (see [12]), we obtain $O(n^2(\sqrt{n} \log n)^{k-3}) = O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ as an upper bound for the number of k-holes in S. □

5.2 A lower bound on the minimum number of (general) k-holes

Every set of k points admits at least one polygonization. Combining this obvious fact with Theorem 6 in [6], we obtain the following result.

Theorem 10. *Let S be a set of n points in the plane in general position. For every $c < 1$ and every $k \leq c \cdot n$, S contains $\Omega(n^2)$ k-holes.*

Proof. We follow the lines of the proof of Theorem 6 in [6]. Consider the point set S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and sets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ of cardinality at least k is

$$\sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^{n} 1 = \frac{(n-k+1)(n-k+2)}{2} = \Theta(n^2).$$

For each $S_{i,j}$ use the $k-2$ points of $S_{i,j} \backslash \{p_i, p_j\}$ which are closest to the segment $p_i p_j$ to obtain a subset of k points including p_i and p_j. Each such set contains at least one k-hole which has p_i and p_j among its vertices. Moreover, as p_i and p_j are the left and rightmost points of $S_{i,j}$, they are also the left and rightmost points of this k-hole. This implies that any k-hole of S can count for at most one set $S_{i,j}$, which gives a lower bound of $\Omega(n^2)$ for the number of k-holes in S. □

A notion similar to k-holes is that of islands. An island I is a subset of S, not containing points of $S \setminus I$ in its convex hull. A k-island is an island of k elements. For example, any two points form a 2-island, and any 3 points spanning an empty triangle are a 3-island of S. In particular, convex k-holes are also k-islands while non-convex k-holes need not be k-islands. In general, any k-tuple spans at most one k-island, while it might span many k-holes. In [21] it was shown that the number of k-islands of S is always $\Omega(n^2)$ and that the Horton set of n points contains only $O(n^2)$ k-islands (for any constant $k \geq 4$ and sufficiently large n).

In contrast to this tight quadratic lower bound on the number of islands, all (families of) point sets we considered so far contain a super-quadratic number of k-holes for any constant $k \geq 4$. For example, the Horton set of n points contains $\Omega(n^3)$ such holes. Also, at first glance, the result from Theorem 10 seems easy to improve. However, note that a general super-quadratic lower bound for the number of k-holes for any constant k would solve a conjecture of Bárány in the affirmative, showing that every point set contains a segment that spans a super-constant number of 3-holes; see e.g. [13, Chapter 8.4, Problem 4]. This might also be a first step towards proving a quadratic lower bound for the number of convex 5-holes. So far, not even a super-linear bound is known for the latter problem [13, Chapter 8.4, Problem 5].

The results in Section 5.3 imply that if there exist point sets with only a quadratic number of 4-holes, then they cannot have a grid-like structure. In particular, the currently best known configuration for minimizing the numbers of convex k-holes, the before-mentioned squared Horton set, cannot serve as an example. Due to this and several other observations we state the following conjecture.

Conjecture 11. *For any constant $k \geq 4$, any n-point set in general position contains $\omega(n^2)$ general k-holes.*

5.3 A lower bound on the number of general k-holes in any perturbed grid

As in Section 5.1, we first deal with the regular integer grid and then derive results for perturbed grids. We again consider an integer grid G of size $\sqrt{n} \times \sqrt{n}$. As a distance measure we use the L_∞-norm, i.e., the distance of two points $p, q \in G$ is the maximum of the differences of their x- and y-coordinates. The length of a segment e spanned by two points of G is defined by the distance of its endpoints. Similar to prime segments, we denote a k-gon in G where all edges are prime segments of G as a *prime k-gon.* Further, let $\square = \partial\,\mathrm{CH}(G)$ be the square forming the boundary of the convex hull of G. We denote a line l that is spanned by points of G and intersects $\square$ on opposite sides, i.e., l either intersects both vertical segments of $\square$ or l intersects both horizontal segments of $\square$, as *cutting line (of G).*

Observation 12. *For an integer grid G of size $\sqrt{n} \times \sqrt{n}$, consider a prime segment pq spanned by two points of G and its supporting line l. If pq has length d, then l contains $O(\sqrt{n}/d)$ points of G. If in addition, l is a cutting line of G, then l contains at least $\lfloor \sqrt{n}/d \rfloor = \Omega(\sqrt{n}/d)$ points of G.*

Theorem 13. *The number of prime 4-holes contained in an integer grid G of size $\sqrt{n} \times \sqrt{n}$ is $\Omega(n^2 \log n / \log\log n)$.*

Proof. Assume that $\sqrt{n} = 3m$ for some integer $m > 0$. Consider contiguous subgrids of G with size $\sqrt{n}/3 \times \sqrt{n}/3$, and denote by C the central such subgrid; see Figure 7. For a fixed point $p \in C$, consider a point $q \in G$ such that pq is a prime segment with length $1 \leq d < \sqrt{n}/3$, and its supporting line l. Let l' and l'' be the two lines parallel to l and spanned by points of G for which no point of G lies between l and l', and between l and l'', respectively. Then for each $l^* \in \{l', l''\}$, there exist two disjoint subgrids of G (both of size $\sqrt{n}/3 \times \sqrt{n}/3$), for which l^* is a cutting line; see again Figure 7 for subgrids where l' is a cutting line. Thus, due to Observation 12, each of the lines l' and l'' contains at least $2 \cdot \left\lfloor \frac{\sqrt{n}}{3d} \right\rfloor = \Omega\left(\frac{\sqrt{n}}{d}\right)$ points of G. As any point of $G \cap \{l' \cup l''\}$ spans a prime triangle with pq (a triangle where all edges are prime segments), pq is the diagonal of $\Omega\left(\frac{n}{d^2}\right)$ prime 4-holes.

Now consider a fixed point $p \in C$ and a distance $1 \leq d < \sqrt{n}/3$. Then the number of points in G that form prime segments with p can be expressed in terms of Euler's phi-function[1]. As $d < \sqrt{n}/3$, this number is exactly $8 \cdot \varphi(d)$. Thus, summing up over all possible distances, we obtain a lower bound of

$$\sum_{d=1}^{\lfloor \sqrt{n}/3 \rfloor - 1} 8 \cdot \varphi(d) \cdot \Omega\left(\frac{n}{d^2}\right) \quad = \quad \Omega\left(n \cdot \sum_{d=1}^{\lfloor \sqrt{n}/3 \rfloor - 1} \frac{\varphi(d)}{d^2}\right)$$

[1] Euler's phi-function, also called Euler's totient function, $\varphi(d)$ denotes the number of positive integers less than d that are relatively prime to d. See for example [28] (page 52). This is the same as the number of segments with endpoints $(0, 0)$ and (a, d), $1 \leq a \leq d$, that do not contain any point with integer coordinates in their interior.

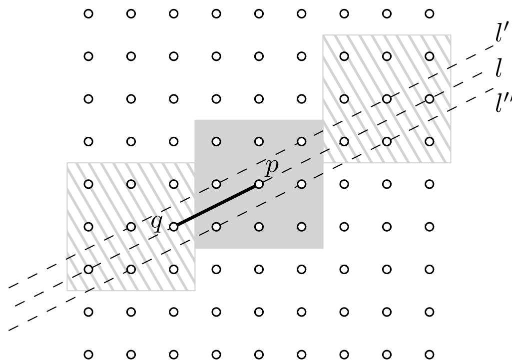

Figure 7: A prime segment pq with $p \in C$, and the according lines l, l', and l''. The central subgrid C is drawn in solid gray. Two disjoint subgrids for which l' is a cutting line are drawn with stripes.

for the number of prime 4-holes that have p as a vertex. Summing up over all $n/9$ points of C, and considering that one 4-hole might have been counted four times, namely once for each of its vertices, we obtain a lower bound for the total number of prime 4-holes in G of

$$\Omega\left(n^2 \cdot \sum_{d=1}^{\lfloor\sqrt{n}/3\rfloor-1} \frac{\varphi(d)}{d^2}\right).$$

Together with the lower bound of $\varphi(d) \geq d/(e^\gamma \log\log d + \frac{3}{\log\log d})$ for $d \geq 3$ where γ is Euler's constant [11] (Theorem 8.8.7) (and $\varphi(d) = 1$ for $d \leq 2$), this implies that the number of prime 4-holes in G can be bounded by

$$\begin{aligned}
\Omega\left(n^2 \cdot \sum_{d=1}^{\lfloor\sqrt{n}/3\rfloor-1} \frac{\varphi(d)}{d^2}\right) &= \Omega\left(n^2 \cdot \sum_{d=3}^{\lfloor\sqrt{n}/3\rfloor-1} \frac{1}{d\cdot(e^\gamma \log\log d + \frac{3}{\log\log d})}\right) \\
&= \Omega\left(\frac{n^2}{e^\gamma \log\log n + 3} \cdot \sum_{d=3}^{\lfloor\sqrt{n}/3\rfloor-1} \frac{1}{d}\right) \\
&= \Omega\left(\frac{n^2 \log n}{\log\log n}\right)
\end{aligned}$$

□

For every constant $k \geq 4$, a prime 4-hole can be extended to a prime k-hole by adding nearby points of the grid. As on the other hand, one prime k-hole contains only a constant number of prime 4-holes, we have the following corollary.

Corollary 14. *For any constant $k \geq 4$, the integer grid G of size $\sqrt{n} \times \sqrt{n}$ contains $\Omega(n^2 \log n / \log\log n)$ prime k-holes.*

In contrast to the bound in Theorem 9, the bound in Corollary 14 is independent of k. The following alternative bound again depends on k.

Theorem 15. *For every $c < 1$ and every $3 \leq k \leq c \cdot 2\sqrt{n}$, the integer grid G of size $\sqrt{n} \times \sqrt{n}$ contains $\Omega(n^{\lfloor k/2 \rfloor/2+1})$ prime k-holes.*

Proof. Consider a k-hole H for which the following properties hold: (i) H is spanned by points in consecutive rows of G, (ii) the lowest of these rows contains exactly one point of H, (iii) the highest of these rows contains one or two points of H, and (iv) all rows in-between contain exactly two points of H that are consecutive in the row (but in general not consecutive along the boundary of H). Figure 8 shows examples of such k-holes for $k \in \{6, 7, 8\}$.

Clearly, all such k-holes are prime. Further, any such k-hole contains points from $\lfloor \frac{k-2}{2} \rfloor + 2 = \lfloor \frac{k}{2} \rfloor + 1$ rows of G. This gives $\sqrt{n} - \lfloor \frac{k}{2} \rfloor$ possible rows for the lowest point of the k-hole. Further, in every row

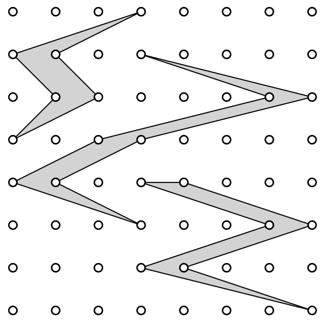

Figure 8: Examples of prime k-holes for the proof of Theorem 15.

there are $\sqrt{n}$ or $\sqrt{n}-1$ possibilities to choose the one or two points for the k-hole, respectively. This gives a total of at least

$$\left(\sqrt{n} - \left\lfloor \frac{k}{2} \right\rfloor\right) \cdot (\sqrt{n}-1)^{\left(\left\lfloor \frac{k}{2}+1 \right\rfloor\right)}$$

such k-holes in G. For every $c < 1$ and every $3 \leq k \leq c \cdot 2\sqrt{n}$, the first factor is $\Omega(\sqrt{n})$. This implies a lower bound of $\Omega(\sqrt{n}^{\lfloor k/2 \rfloor + 2}) = \Omega(n^{\lfloor k/2 \rfloor/2+1})$ on the number of these k-holes and thus on the total number of k-holes in G. □

When comparing the two lower bounds provided in Corollary 14 and Theorem 15, we observe that the bound from Theorem 15 beats the one from Corollary 14 for $k \geq 6$, while Corollary 14 gives a better bound for $k \in \{4, 5\}$.

As a last result of this section, we translate these bounds from the integer grid G to grid-like sets of points in general position (for example squared Horton sets). To this end, let an ε-perturbation $p_\varepsilon(G)$ of G be a perturbation where every point of G is replaced by a point at distance at most ε. Observe that there exists an $\varepsilon > 0$ such that for any ε-perturbation $p_\varepsilon(G)$ of G, every prime k-hole in G is also a k-hole in $p_\varepsilon(G)$. This is not true (not even for arbitrarily small ε) for a non-prime k-hole in G, i.e., a k-hole having on its boundary points of G that are not vertices of that k-hole. As in Corollary 14 and Theorem 15 we count only prime k-holes, we obtain the following statement.

Corollary 16. *There exists an $\varepsilon > 0$ such that any ε-perturbation $p_\varepsilon(G)$ of an integer grid G of size $\sqrt{n} \times \sqrt{n}$ contains*

$\Omega(n^2 \log n / \log\log n)$ *4-holes,*

$\Omega(n^2 \log n / \log\log n)$ *5-holes, and*

$\Omega(n^{\lfloor k/2 \rfloor/2+1})$ *k-holes for any $6 \leq k \leq c \cdot 2\sqrt{n}$ with $c < 1$.*

6 Conclusion

We have shown various lower and upper bounds on the numbers of convex, non-convex, and general k-holes and k-gons in point sets. Several questions remain unsettled, where the maybe most intriguing open question is to prove Conjecture 11, i.e., to show a super-quadratic lower bound for the number of general k-holes for $k \geq 4$.

Acknowledgments. Research on this topic was initiated during the *Third Workshop on Discrete Geometry and its Applications* in Morelia (Michoacán, Mexico). We thank Edgar Chávez and Feliu Sagols for helpful discussions. An extended abstract of part of this article has been presented in [5].

References

[1] B. M. Ábrego, M. Cetina, S. Fernández-Merchant, J. Leaños, and G. Salazar. On ($\leq k$)-edges, crossings, and halving lines of geometric drawings of K_n. *Discrete and Computational Geometry*, (48):192–215, 2012.

[2] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar. A central approach to bound the number of crossings in a generalized configuration. *Electronic Notes in Discrete Mathematics*, 30:273–278, 2008.

[3] O. Aichholzer. [Empty] [colored] k-gons - Recent results on some Erdős-Szekeres type problems. In *Proc. EGC2009*, pages 43–52, Zaragoza, Spain, 2009.

[4] O. Aichholzer. On the rectilinear crossing number. http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/, 2013.

[5] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber. On k-gons and k-holes in point sets. In *Proc. CCCG'11*, pages 21–26, Toronto, Canada, 2011.

[6] O. Aichholzer, R. Fabila-Monroy, H. Gonzlez-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. *CGTA*, 47(6):644 – 650, 2014. special issue of EuroCG'11.

[7] O. Aichholzer, R. Fabila-Monroy, T. Hackl, C. Huemer, A. Pilz, and B. Vogtenhuber. Lower bounds for the number of small convex k-holes. *CGTA*, 47:605–613, 2014.

[8] O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-gons and 5-holes. In *Proc. XIV Encuentros de Geometría Computacional EGC2011*, pages 7–10, Alcalá de Henares, Spain, 2011.

[9] O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-gons and 5-holes. *LNCS*, 7579:1–13, 2012. Special issue of EGC2011.

[10] O. Aichholzer and H. Krasser. The point set order type data base: A collection of applications and results. In *Proc. CCCG'01*, volume 13, pages 17–20, 2001.

[11] E. Bach and J. Shallit. *Algorithmic Number Theory (Vol I: Efficient Algorithms)*. MIT Press, Cambridge, MA, 1996.

[12] I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons. *Studia Scientiarum Mathematicarum Hungarica*, 41(2):243–266, 2004.

[13] P. Brass, W. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, 2005.

[14] K. Dehnhardt. *Leere konvexe Vielecke in ebenen Punktmengen*. PhD thesis, TU Braunschweig, Germany, 1987. In German.

[15] A. Dumitrescu. Planar sets with few empty convex polygons. *Studia Scientiarum Mathematicarum Hungarica*, 36(1-2):93–109, 2000.

[16] A. Dumitrescu, A. Schulz, A. Sheffer, and C. Tóth. Bounds on the maximum multiplicity of some common geometric graphs. In *Proc. 28th International Symposium on Theoretical Aspects of Computer Science (STACS 2011)*, Leibniz International Proceedings in Informatics (LIPIcs), pages 637–648, Dagstuhl, Germany, 2011.

[17] P. Erdős. Some more problems on elementary geometry. *Australian Mathematical Society Gazette*, 5:52–54, 1978.

[18] P. Erdős and R. Guy. Crossing number problems. *The American Mathematical Monthly*, 88:52–58, 1973.

[19] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[20] P. Erdős and G. Szekeres. On some extremum problems in elementary geometry. *Ann. Univ. Sci. Budapest. Eötvös, Sect. Math.*, 3/4:53–62, 1960.

[21] R. Fabila-Monroy and C. Huemer. Covering islands in plane point sets. *LNCS*, 7579:220–225, 2012. Special issue of EGC2011.

[22] R. Fabila-Monroy and J. López. Computational search of small point sets with small rectilinear crossing number. *Journal of Graph Algorithms and Applications*, 18(3):393–399, 2014.

[23] A. García. A note on the number of empty triangles. In *Proc. XIV Encuentros de Geometría Computacional ECG2011*, pages 101–104, Alcalá de Henares, Spain, 2011.

[24] A. García. A note on the number of empty triangles. *LNCS*, 7579:249–257, 2012. Special issue of ECG2011.

[25] A. García, M. Noy, and J. Tejel. Lower bounds on the number of crossing-free subgraphs of K_n. *CGTA*, 16:211–221, 2000.

[26] T. Gerken. Empty convex hexagons in planar point sets. *Disc. Comp. Geom.*, 39(1–3):239–272, 2008.

[27] H. Harborth. Konvexe Fünfecke in ebenen Punktmengen. *Elemente der Mathematik*, 33:116–118, 1978. In German.

[28] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, London, fifth edition, 1979.

[29] J. Horton. Sets with no empty convex 7-gons. *Canadian Mathematical Bulletin*, 26(4):482–484, 1983.

[30] J. Kalbfleisch, J. Kalbfleisch, and R. Stanton. A combinatorial problem on convex n-gons. In *Proc. Louisiana Conference on Combinatorics, Graph Theory and Computing, 1970*, pages 180–188, Louisiana State University.

[31] V. A. Koshelev. On Erdős–Szekeres problem for empty hexagons in the plane. *Model. Anal. Inform. Sist*, 16(2):22–74, 2009. In Russian.

[32] J. A. D. Loera, J. Rambau, and F. Santos. *Triangulations: Structures for Algorithms and Applications*, volume 25 of *Algorithms and Computation in Mathematics*. Springer-Verlag, 2010.

[33] C. Nicolás. The empty hexagon theorem. *Disc. Comp. Geom.*, 38(2):389–397, 2007.

[34] R. Pinchasi, R. Radoičić, and M. Sharir. On empty convex polygons in a planar point set. *J. Comb. Theory Ser. A*, 113:385–419, April 2006.

[35] M. Sharir, A. Sheffer, and E. Welzl. Counting plane graphs: Perfect matchings, spanning cycles, and Kasteleyn's technique. *J. Comb. Theory Ser. A*, 120:777–794, 2013.

[36] G. Szekeres and L. Peters. Computer solution to the 17-point Erdős–Szekeres problem. *The ANZIAM Journal*, 48(2):151–164, 2006.

[37] G. Tóth and P. Valtr. The Erdős–Szekeres theorem: upper bounds and related results. *Combinatorial and Computational Geometry, J.E. Goodman, J. Pach, and E. Welzl (Eds.),*, 52:557–568, 2005.

[38] P. Valtr. Convex independent sets and 7-holes in restricted planar point sets. *Disc. Comp. Geom.*, 7:135–152, 1992.

[39] P. Valtr. On the minimum number of empty polygons in planar point sets. *Studia Scientiarum Mathematicarum Hungarica*, 30:155–163, 1995.

[40] P. Valtr. On empty hexagons. *J. E. Goodman, J. Pach, and R. Pollack, Surveys on Disc. Comp. Geom., Twenty Years Later, Contemp. Math. 453, AMS*, 453:433–441, 2008.

[41] P. Valtr. On empty pentagons and hexagons in planar point sets. In *Proc. CATS2012*, pages 47–48, Melbourne, Australia, 2012.

[42] B. Vogtenhuber. *Combinatorial Aspects of [Colored] Point Sets in the Plane*. PhD thesis, Institute for Software Technology, Graz University of Technology, Graz, Austria, 2011.