On the lenght of Longest Alternating Paths for Multicolored Point Sets in Convex Position

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Abstract

Let P be a set of points in $\mathbb{R}^2$ in general position such that each point is coloured with one of k colours. An alternating path of P is a simple polygonal whose edges are straight line segments joining pairs of elements of P with different colors. In this paper we prove the following: Suppose that each colour class has cardinality s and P is the set of vertices of a convex polygon. Then P always has an alternating path with at least $(k-1)s$ elements. Our bound is sharp for odd values of k.

1 Introduction

Let P be a collection of $2s$ points in general position on the plane. Suppose that s elements of P are colored red, and s blue. An alternating path of P is a simple polygonal whose edges are straight line segments joining pairs of elements of P with different colors, see Figure 1. Alternating paths of point sets were first studied in Akiyama and Urrutia [3]. In that paper, an algorithm that decides if an alternating path that covers all the elements of P exists is given when the elements of P are in convex position, i.e. the elements of P are the vertices of a convex polygon.

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In [1] they study the problem of finding an alternating path for a point set in general position. They show that if all the red elements of P are separated from all the blue elements by a line or if all the blue points are contained in the convex hull of the red points, then there is an alternating path that covers all the elements of P. Later, in [2], this result is used this to prove that any point set P in general position always has an alternating path that covers at least half of the elements of P. There it is also proved that there are point sets in convex position such that any alternating path covers at most $\frac{2}{3}$ of the elements of P and it is conjectured that this bound is sharp.

Here we consider the following generalization of the later problem. Let P be a point set in convex position with $3s$ elements s red, s blue, and s black. We show that P always admits an alternating path, defined as before, that covers $2s$ elements of P and that this bound is sharp. In general we show that if P has ks points, and for each $1 \leq i \leq k$ it has s points colored i, then P admits an alternating path of length at least $(k-1)s$, and this bound is sharp for odd values of k. In Section 2 we prove our main tool for analyzing the length of alternating paths in a collection of points P in convex position. We then return to the problem stated in this introduction in Section 3. We revise the case for two colours in Section 4 and in Section 5 we give some final remarks and some problems.

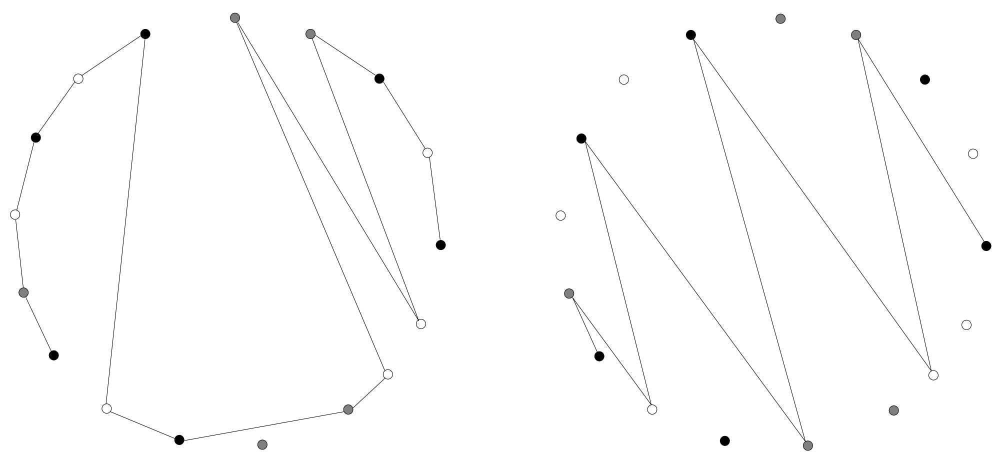

Figure 1: On the left-hand side, we have an alternating path and on the right-hand side we have a zig-zag path, a particular type of alternating path.

2 Point sets with k colours

Let $P_1, \ldots, P_k$, $k > 1$, be a collection of disjoint non-empty point sets such that the points in $P = P_1 \cup \ldots \cup P_k$ are in convex position and $|P| = n$. We consider that points in P_i have colour c_i and that $c(u)$ is the colour of the point $u \in P$. An alternating path of points in P is a simple polygonal whose edges are straight line segments joining elements of P with different colour, see Figure 1. We assume that $|P_i| \geq 2$.

An alternating path Z of P will be called a *zig-zag path* if there is a line l that intersects all the edges of Z, see Figure 1 for an example. If $|P_1| = \ldots = |P_k|$, P will be called a *k-balanced point set.*

Suppose that the elements of P are labelled with the integers $\{0, \ldots, n-1\}$ such that consecutive points in the convex hull of P receive consecutive integers (assuming that $n-1$ and 0 are consecutive). We construct the zig-zag path Z as follows:

- The first vertex of Z is 0.
- Let i_1 be the smallest integer such that $c(i_1) \neq c(0)$. The second vertex of Z is i_1.
- Let j_1 be the largest integer such that $c(j_1) \neq c(i_1)$. the point j_1 is the third vertex of Z.
- Suppose that the first $2k+1$ (respectively $2k+2$) vertices of Z, 0, i_1, $j_1, \ldots, i_k$, j_k have been chosen. Then the next vertex of Z corresponds to the smallest integer i_{k+1}, if exists, such that $c(i_{k+1}) \neq c(j_k)$ and $i_k < i_{k+1} < j_k$ (respectively, the largest j_{k+1}, if exists, such that $c(i_{k+1}) \neq c(j_{k+1})$ and $i_{k+1} < j_{k+1} < j_k$).

Now, for a given element u of P, we define two zig-zag paths for it. The first one Z_u^+ is obtained by the above procedure when relabelling the elements of P in the *clockwise* direction with the integers $\{0, \ldots, n-1\}$ starting from u. The second one Z_u^- is obtained in the same way but we relabel in an *counterclockwise* direction starting from u. See Figure 2 for an example. Here we consider that two alternating paths are the same if they have the same set of segments.

Remark 2.1. *Z_u^+ and Z_u^- are different zig zag paths.*

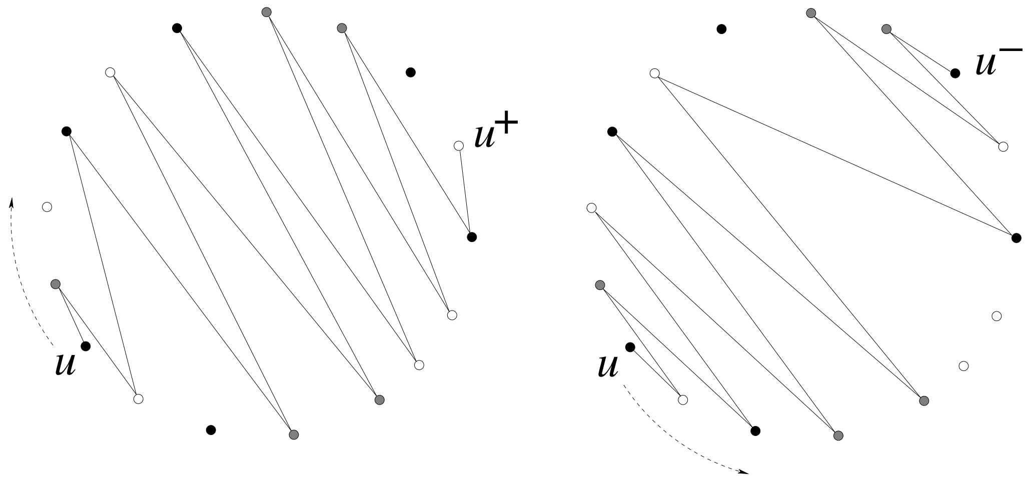

Figure 2: On the left-hand side we have Z_u^+, the zig-zag path in the clockwise direction starting from u; and u^+, the antipode of u with respect to Z_u^+. On the right-hand side we have Z_u^- and u^-.

The zig-zag path Z_u^+ has two endpoints, u itself and a unique point $v = u^+$ different from u, which we called the antipode of u with respect to Z_u^+. During the procedure to define Z_u^+, u^+ receives a label i and its (only) neighbour w in Z_u^+ a label j. If $i < j$, the $Z_u^+ = Z_{u^+}^+$. If $i > j$, then $Z_u^+ = Z_{u^+}^-$. We denote this unique zig-zag path $Z_{u^+}^{\pm}$. We define similarly $Z_{u^-}^{\pm}$. See Figure 3 for an example.

Remark 2.2. *$Z_{u^+}^{\pm}$ (respectively $Z_{u^-}^{\pm}$) is the same as either Z_w^+ or Z_w^- but not both, for some unique $w \in P$ different from u.*

Consider the set $\mathcal{Z} = \{Z_u^+, Z_u^- \,|\, u \in P\}$ of zig-zag paths. We have the following lemmas.

Lemma 2.3. *The cardinality of $\mathcal{Z}$ equals $|P|$.*

Proof. Construct the graph G on $V = \mathcal{Z} \cup P$ by adding all the edges $\{u, Z_u^+\}$ and $\{u, Z_u^-\}$ for all $u \in P$. Thus G is a bipartite graph. Also the degree of any u in P is 2, by Remark 2.1. And, by the Remark 2.2, the degree of any Z in $\mathcal{Z}$ is also 2. So G is a 2-regular bipartite graph and, by the *marriage theorem*, see [4], $|P| = |\mathcal{Z}|$. □

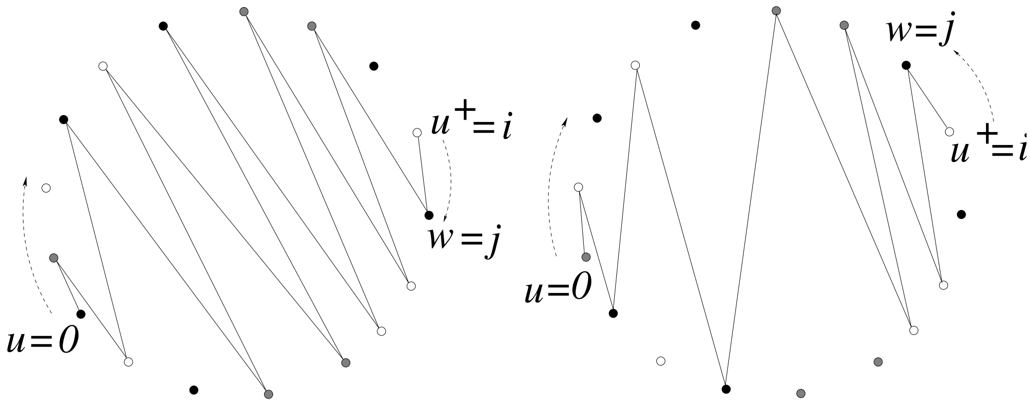

Figure 3: On the left-hand side we have the case when Z_u^+ is also a Z_w^+ for some $w \neq u$. We denote this unique Z_w^+ by $Z_{u^+}^{\pm}$. On the right-hand side we have the case when Z_u^+ is a Z_w^- for some $w \neq u$. Similarly, we denote this path by $Z_{u^-}^{\pm}$.

Lemma 2.4. *For any given segment $\overline{uv}$ with endpoints u and v with different colours, there are exactly two elements of $\mathcal{Z}$ that use $\overline{uv}$.*

Proof. Let u and v be points in P with different colours. The segment $\overline{uv}$ splits P into two sets of points P_1 and P_2 in convex position such that $P_1 \cap P_2 = \{u, v\}$.

In P_1, u and v are next to each other. Suppose w.l.o.g. that in P_1, v follows u in the clockwise direction. Lets take $Z_1 = Z_u^+$ in P_1 and $Z_2 = Z_v^+$ in P_2. The segment $\overline{uv}$ is the only one that is shared by Z_1 and Z_2. Lets take $Z_{u^+}^{\pm}$ in P_1 as constructed before. By Remark 2.2, we can suppose w.l.o.g. that $Z_{u^+}^{\pm} = Z_w^+$ for some unique $w \in P_1$. Clearly, Z_w^+ in P is precisely the zig-zag path that consists of the segments of $Z_1 \cup Z_2$.

In a similar fashion, $Z_1' = Z_v^-$ in P_1 and $Z_2' = Z_u^-$ in P_2 define a unique zig-zag path of the form either $Z_{w'}^+$ or $Z_{w'}^-$ in P that consists of the segments of $Z_1' \cup Z_2'$. As at least one of P_1 or P_2 are not empty, we have that $Z_1 \neq Z_1'$ or $Z_2 \neq Z_2'$. Thus the two paths defined above are different.

Finally, lets suppose that there exists $Z = Z_{w''}^+$ such that $\overline{uv}$ belongs to Z. Suppose w.l.o.g. that $w'' \in P_1$ and that u is before v when traversing P in the clockwise direction starting from w''.

We have two cases. If v is before u in Z when starting from w'', then $Z = Z_1 \cup Z_2$, where $Z_1 = Z_u^+$ in P_1 and $Z_2 = Z_v^+$ in P_2. Else, $Z = Z_1' \cup Z_2'$, where $Z_1' = Z_v^-$ in P_1 and $Z_2' = Z_u^-$ in P_2. In both cases Z was already constructed. The case when $Z = Z_{w''}^-$ is similar.

We conclude that for each segment $\overline{uv}$ with endpoints coloured different, there are exactly two elements of $\mathcal{Z}$ that use it. □

We compute now $\sum_{Z \in \mathcal{Z}} l(Z)$, where $l(Z)$ is the length (number of segments) of Z.

Proposition 2.5. *Let $P_1, \ldots, P_k$ be a collection of points such that the points in $P_1 \cup \ldots \cup P_k$ are in convex position. Then there exists a set $\mathcal{Z}$ of zig-zag paths such that*

$$\sum_{Z \in \mathcal{Z}} l(Z) = 2 \sum_{\substack{1 \leq i,j \leq k \\ i \neq j}} n_i n_j,$$

where $n_i = |P_i|$.

Proof. Construct $\mathcal{Z}$ as before. The result follows from Lemma 2.4 as any segment that joins two points of different sets P_i and P_j is in exactly two elements of $\mathcal{Z}$. □

Theorem 2.6. *Let $P_1, \ldots, P_k$ be a collection of points such that $P = P_1 \cup \ldots \cup P_k$ is in convex position and $|P_1| \geq |P_2| \geq \ldots \geq |P_k|$. Then there exists an alternating path of length at least $|P| - |P_1|$.*

Proof. Let $n_i = |P_i|$, $1 \leq i \leq k$, and let $\mathcal{Z}$ be the set of zig-zag paths in P as constructed before. From Proposition 2.5 and Lemma 2.3 the average length of the elements in $\mathcal{Z}$ is

$$\begin{aligned} \frac{1}{|\mathcal{Z}|} \sum_{Z \in \mathcal{Z}} l(Z) &= \frac{1}{n} \sum_{\substack{1 \leq i,j \leq k \\ i \neq j}} 2 n_i n_j \\ &= \frac{1}{n}\left(n^2 - \sum_{i=1}^{k} n_i^2\right) \geq n - n_1. \end{aligned}$$

By the basic principle of the *probabilistic method*, see [6], there exists an element of $\mathcal{Z}$ of length at least $n - n_1$. □

Corollary 2.7. *Let P be a k-balanced point set with ks points. Then there exists an alternating path of length $(k-1)s$.*

Proof. Follows directly from Theorem 2.6. □

3 Point sets with an odd number of colours

From the previous section we know that if we have a $(2r+1)$-balanced point set P with $(2r+1)s$ points, there is an alternating path that covers at least $2rs+1$ points. For the case $r=1$ we have the following example.

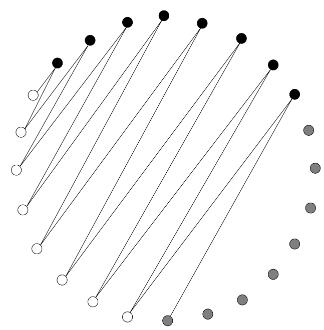

Figure 4: An example of a 3-balanced point set with 24 points and where the maximum alternating path has length 16.

Let P be a point in convex position and with $3s$ points, s red, s blue and s black. Furthermore, the red points are consecutive, the same as the blue and black points. See Figure 4.

Then, any alternating path will cover at most $2s+1$ points. So, our lower bound is tight in the case $r=1$. For $r>1$, lets take P with $n=(2r+1)s$ elements labelled with the integers $\{0,\dots,n-1\}$. We color the point i with colour j if $(j-1)s \leq i < js$, for some $1 \leq j \leq 2r+1$.

Any alternating path induces a planar matching in P, where two points are matched just if they have different colours. The maximum planar matching with this property saturates $(2r)s$ points. One of these maximum matchings is the one with edges $\{i, n-1-i\}$ for $0 \leq i \leq rs-1$. This matching induces a ziz-zag alternating path Z with endpoints 0 and $(r+1)s$. We can extend this alternating path Z to Z' by adding the segment $\overline{(r+1)s, rs}$ but cannot be extended further. Thus, the alternating path Z' is clearly of maximum length and covers $2rs+1$ points of P. We have the following result.

Theorem 3.1. *For any integer $k>1$ odd, there exists a k-balanced point set P with ks points such that any alternating path has length at most $(k-1)s$.*

Therefore, the bound given in Corollary 2.7 is tight.

4 Two colors revisited

In [2] it is proved that any 2-balanced point set in general position admits an alternating path which covers at least half of the elements of P and starts from any given point lying on the convex hull of P. Their proof is based on the following lemma from [1]:

Lemma 4.1. *Let Q be a point set with $2m$ points, m red such that there is a line l that separates the blue from the red elements of Q. Then there is an alternating path that covers all the elements of Q.*

If the points in P are in convex position, the previous result implies that we can find an alternating path which covers at least half of the elements of P and starts from any given point of P. We give another proof of this without using Lemma 4.1.

Theorem 4.2. *Let P be a 2-balanced point set in convex position with $2s$ points, and u be any element of P. Then there is a zig-zag alternating path starting at u that covers at least s elements of P.*

Indeed we will show that there are two zig-zag alternating paths starting at u that together cover all the elements of P. See Figure 5.

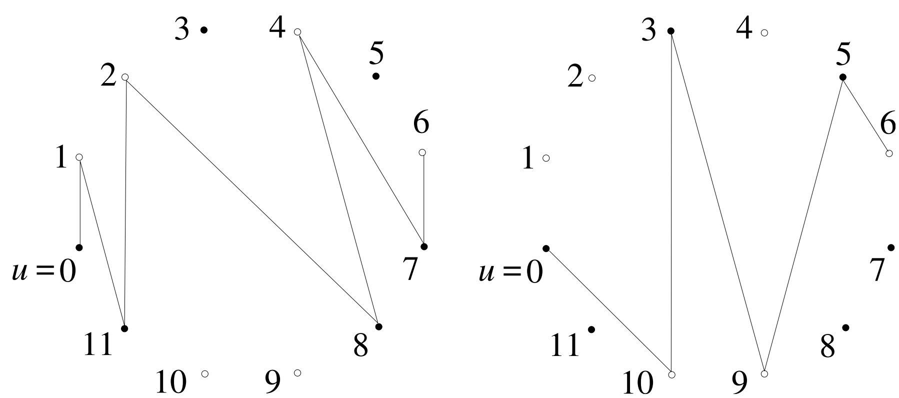

Figure 5: On the left-hand side we have Z_u^+ and on the right-hand side we have Z_u^-. Together they form a cycle that uses al the points.

Lemma 4.3. *$Z_u^+ \cup Z_u^-$ is a cycle that covers all the vertices of P.*

Proof. Let l be the line joining u with its antipode u^+ with respect to Z_u^+. Observe that if this line leaves k red points above it (unused by Z_u^+), it leaves exactly $k+1$ blue points below it (unused by Z_u^+). Together with the first and last vertices of Z_u^+ these points are the ones used by Z_u^-. Our lemma follows. □

The proof of Theorem 4.2 follows immediately.

5 Conclusions

If P is $2r$-balanced, deciding if P has an alternating path which covers P can be done in $O(|P|^2)$ by using a similar algorithm as in [3]. In general, finding the maximum length of an alternating path in P can be done also in $O(|P|^2)$ by dynamic programming as in [2].

Let P be a k-balanced point set with $n = ks$ points. If k is odd, we have shown that the lower bound $(k-1)s$ for the length of an alternating path on P is tight. However the case k even appears to be more difficult and even the case $k = 2$ has not been settled. The results in [2] indicate that probably the right value for the lower bound is $\frac{2}{3}n$. For k even, $k > 2$, our results give a lower bound of $\frac{k-1}{k}n$, but further improvement could be expected in this case.

The case when P is a set of points in general position is almost unexplored. Some results are given in [2, 5] when P is coloured with 2 colours.

References

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