Balanced 6-holes in bichromatic point sets*

Oswin Aichholzer† Jorge Urrutia‡ Birgit Vogtenhuber†

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Abstract

We consider an Erdős type question on k-holes (empty k-gons) in bichromatic point sets. For a bichromatic point set $S = R \cup B$, a balanced $2k$-hole in S is spanned by k points of R and k points of B. We show that if $|R| = |B| = n$, then the number of balanced 6-holes in S is at least $\frac{1}{45}n^2 - \Theta(n)$.

1 Introduction

Let $S = R \cup B$ be a bichromatic point set in the plane (where R is the set of red, and B is the set of blue points) in general position, i.e., no three points of S lie on a common straight line. A *k-gon (in S)* is a simple polygon P spanned by k vertices of S. If P does not contain any points of S in its interior, it is called a *k-hole*. A *$2k$-hole* with k red and k blue vertices is called *balanced*.

Background. The study of k-gons in point sets goes back to Erdős [6]. Since then, many variants have been considered, see e.g. [1] for a survey. The topic of holes in colored point sets was introduced by Devillers et al. [5], who studied the existence of monochromatic convex k-holes in l-colored point sets.

Recently, Bereg et al. [4] considered balanced 4-holes in bichromatic point sets $S = R \cup B$. They give a characterization of point sets containing convex balanced 4-holes and present a class of linearly separable sets with $|R| = |B| = n$ that do not contain any convex balanced 4-hole. Further, they show that if $|R| = |B| = n$, then S contains a quadratic number of balanced, not necessarily convex, 4-holes.

In this work we show that the same is true for balanced 6-holes. In [3] we studied the same question for the special case where the color classes R and B are linearly separable.

Theorem 1. *The number of balanced 6-holes in bichromatic point sets $R \cup B$ with $|R| = |B| = n$ is at least $\frac{1}{45}n^2 - \Theta(n)$.*

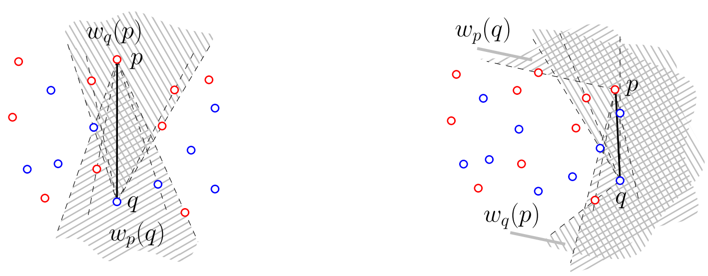

Figure 1: Examples of p-wedges and q-wedges.

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†Institute for Software Technology, University of Technology, Graz, Austria, [oaich|bvogt]@ist.tugraz.at

‡Instituto de Matemáticas, Universidad Nacional Autónoma de México, D.F. México, México, urrutia@matem.unam.mx

Definitions and notation. Let $S = R \cup B$ be a point set in general position in the plane with $|R| = |B| = n \geq 3$. For an arbitrary point $p \in S$, consider the remaining set $S \backslash \{p\}$ in circular order around p. We denote this (cyclically) sorted set by S_p.

Let $K(R, B)$ be the complete bipartite graph of S, i.e., the graph which contains an edge pq for every pair of vertices p, q with $p \in R$, $q \in B$. For an edge pq of $K(R, B)$, we denote the two preceeding and the two succeeding points of q in S_p as *p-neighbors (of q)*. Further, we denote by $r_p(q)$ and $b_p(q)$ the number of red and blue p-neighbors of q, respectively. Trivially, $0 \leq r_p(q), b_p(q) \leq 4$ and $r_p(q) + b_p(q) = 4$. Finally, we denote the closed region which is spanned by the rays from p to the p-neighbors of q and contains q and its neighbors as *p-wedge (of q)*, $w_p(q)$. Note that $w_p(q)$ contains exactly p, q, and the p-neighbors of q. Moreover, $w_p(q)$ might be convex or non-convex; see Figure 1. Similarly, let the *extended p-wedge of q* $w_p^e(q)$ be the closed region which is spanned by the rays from p to the three succeeding and preceeding points of q in S_p, respectively, that contains q. We define a 5-coloring for the edges of $K(R, B)$ in the following way, where the decision on which color an edge obtains is taken in the given order.

The coloring scheme. (1) Edges pq where at least one of the extended wedges $w_p^e(q)$ and $w_q^e(p)$ is non-convex are colored gray. (2) Edges pq for which there exists a balanced 6-hole having both p and q as vertices are colored green. (3) Edges pq with $r_p(q) + r_q(p) \geq 7$ are colored red. (4) Edges pq with $b_p(q) + b_q(p) \geq 7$ are colored blue. (5) All remaining edges are colored black.

Theorem 1 follows from the fact that the number of green edges is quadratic. In order to show this, we prove that (1) the number of gray edges is linear, (2) there are in fact no balck edges, and (3) the number of blue and red edges is at most $\frac{2}{5}n^2 + \Theta(n)$ each.

As an upper bound for the minimum number of balanced 6-holes, $O(n^{7/2} \log^3 n)$ follows immediately from the uncolored case ([2], Thm. 6).

References

[1] O. Aichholzer. [Empty] [colored] k-gons - Recent results on some Erdős-Szekeres type problems. In *Proc. ECG2009*, pages 43–52, Zaragoza, Spain, 2009.

[2] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber. On k-gons and k-holes in point sets. In *Proc. CCCG'11*, pages 21–26, Toronto, Canada, 2011.

[3] O. Aichholzer, J. Urrutia, and B. Vogtenhuber. Balanced 6-holes in linearly separable bichromatic point sets. In *Proc. VII Latin-American Algorithms, Graphs, and Optimization Symposium LAGOS 2013*, pages 229–234, Playa del Carmen, Quintana Roo, Mexico, 2013.

[4] S. Bereg, J. M. Díaz-Báñez, R. Fabila-Monroy, P. Pérez-Lantero, A. Ramírez-Vigueras, T. Sakai, J. Urrutia, and I. Ventura. On balanced 4-holes in bichromatic point sets. In *Proc. EuroCG'12*, pages 201–201, Assisi, Italy, 2012.

[5] O. Devillers, F. Hurtado, G. Károlyi, and C. Seara. Chromatic variants of the Erdős-Szekeres Theorem. *CGTA*, 26(3):193–208, 2003.

[6] P. Erdős. Some more problems on elementary geometry. *Austral. Math. Soc. Gaz.*, 5:52–54, 1978.