A Note on Balanced Colourings
for Lattice Points

JIN AKIYAMA

Department of Mathematics, Tokai University, Hiratsuka, Japan.

JORGE URRUTIA*

Department of Computer Science, University of Ottawa, Ottawa, Ontario, Canada.

The following problem was posed in the 27th International Mathematics Olympiad (1986):

One is given a finite set of points P_n in the plane, each point having integer coordinates. Is it always possible to colour some of the points red and the remaining points white in such a way that, for any straight line L parallel to either one of the coordinate axes, the difference (in absolute value) between the number of white points and red points on L is not greater than 1?

It is not hard to see that the answer to the above question is "yes". In this note we generalize this result, and show that P_n can be coloured with m ($m \geq 2$) colours in such a way that for any straight line parallel to either one of the coordinate axes, the difference (in absolute value) between the number of points coloured i and the number of points coloured j is at most 1, $1 \leq i < j \leq m$. A conjecture for the higher dimensional case is presented.

Let P_n be a subset of n elements of the lattice points L of $\mathbb{R}^2$, i.e. $(x,y) \in P_n$ iff $x,y \in \mathbb{N}$. For every $I \in \mathbb{N}$, let R_i and C_i be the rows and columns of P_n, i.e. $R_i = \{(x,y) \in P_n : y=i\}$ and $C_i = \{(x,y) \in P_n : x=i\}$. An m-colouring of P_n is a partitioning of P_n into m subsets $S_i,\ldots,S_m$. Given an m-colouring of P_n, let $R_{i,j}=R_i \cap S_j$ and $C_{i,j}=C_i \cap S_j$.

An m-colouring of P_n is called *almost balanced* if for any row or column of $\mathbb{R}^2$, we have: $||R_{i,j}| - |R_{i,k}|| \leq 1$ and $||C_{i,j}| - |C_{i,k}|| \leq 1$. In words, an m-colouring of P_n is almost balanced if for every row and column of $\mathbb{R}^2$ the number of elements coloured j differs from the number of elements coloured k by at most one.

Our main goal in this note is to prove the following result:

Theorem 1: Let $P_n \in L$. Then P_n can always be m-coloured with an almost balanced m-colouring, $1 \leq m \leq n$.

Proof: Split each row R_i of P_n into $\lceil |R_i|/m \rceil$ disjoint subsets $R_i(1), ..., R_i(\lceil |R_i|/m \rceil)$, all of which, save at most one, have exactly m elements, $i=1,..., (\lceil |R_i|/m \rceil$ Similarly split each column C_i into $\lceil |C_i|/m \rceil$ subsets $C_i(1), ..., C_i(\lceil |C_i|/m \rceil)$. (See Figure 1).

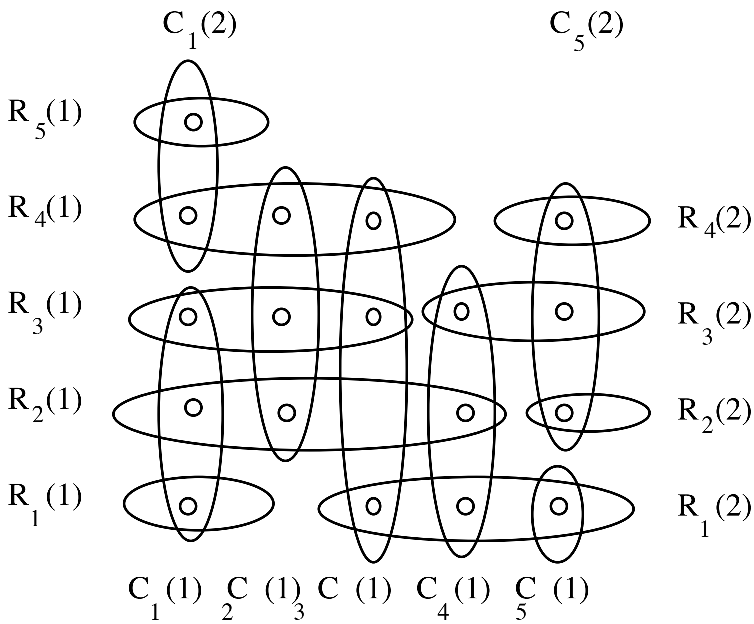

Figure 1.

Construct a bipartite graph H with vertex set $V(H) = \{C_i(j) : j=1,..., \lceil |C_i|/m \rceil\} \cup \{R_i(j) : j=1,..., \lceil |R_i|/m \rceil\}$. Two vertices $R_i(j)$, $C_k(l)$ are adjacent in H if and only if $R_i(j) \cap C_k(l) \neq \emptyset$. (See Figure 2(a)). Since any two $R_i(j)$ and $C_k(l)$ have at most one element in common, and each $p \in P_n$ belongs to exactly one pair of intersecting sets $R_i(j)$, $C_k(l)$, there is a one to one mapping f:$E(H) \rightarrow P_n$ between the edges of H and the elements of P_n.

Moreover since each set $R_i(j)$, $C_k(l)$ has at most m elements, the maximum degree in H is m. Thus by Vizing's Theorem, H is m-edge colourable. However any such colouring of H induces (by using f) an m-colouring of P_n. (See Figure 2a,2b).

It follows now that this m-colouring of P_n is an almost balanced m-colouring of

P_n.

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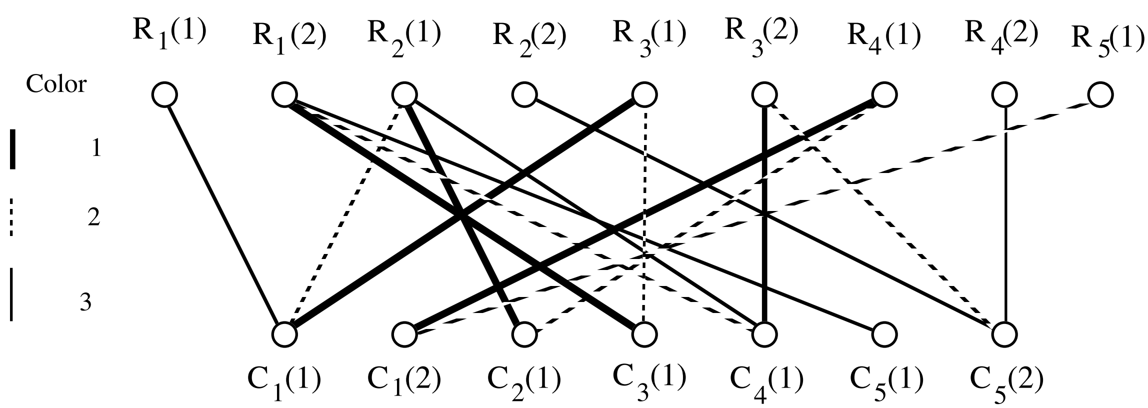


Figure 2a.

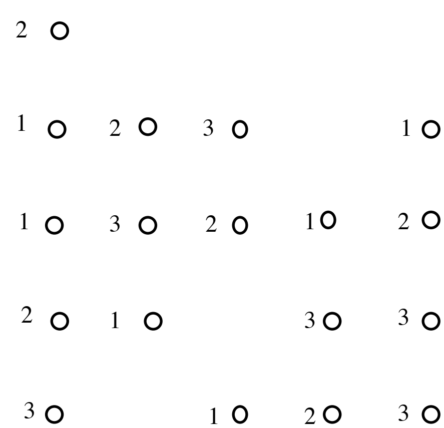


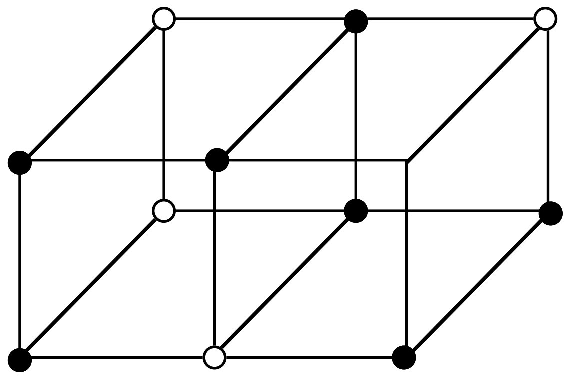
Figure 2b.

Remark 1: Notice that the proof given here gives in a natural way a polynomial time algorithm to find almost balanced m-colourings of P_n. The complexity of such an algorithm equals that of finding m-edge colourings in bipartite graphs.

An interesting question arises:

Is it possible to generalize our result to the case when P_n is a subset of the lattice points in $\mathbb{R}^k$? More specifically, let $P_n \subset \mathbb{R}^k$, $k>2$. For what values of m, if any, can we always find almost balanced m-colourings of P_n ?

The configuration P_7 of 7 lattice points in Figure 3 shows that there are no almost balanced 2-colourings for P_7.



P_7 (black points)

Figure 3.

Conjecture: There exist collections of points P_n in the lattice points of $\mathbb{R}^3$ such that there exists no almost balanced 3-colouring for P_n.