Illuminating Triangles and Quadrilaterals with Vertex Floodlights

Felipe Contreras[1], Jurek Czyzowicz[2], Nicolas Fraiji[1] and Jorge Urrutia[1]

Abstract

We show that three $\pi/6$ vertex floodlights suffice to illuminate any triangle, and we prove that any quadrilateral can be illuminated by three $\pi/4$ vertex floodlights.

1 Introduction

There has been recent interest in illumination by floodlights [1] [2] [7], a new variation on visibility problems in computational geometry. A floodlight of angle α is a light that projects in a (contiguous) cone of angle α. A floodlight placed in a simple polygon with it's apex at a vertex is called a *vertex light*; if it has an angle α, it will be called an α- *vertex light*. The question of illuminating a polygon with vertex floodlights was first explored in [3], in which it was established that every orthogonal polygon can be covered by $\lfloor 3(n-1)/8 \rfloor$ $\pi/2$ vertex lights. This work raised various questions and problems, one of them was the question of what vertex floodlights suffice to cover any convex polygon of n vertices. Urrutia established that a convex polygon may be illuminated by any three vertex floodlights whose total angle is π.
Another question was the following question posed as an open problem in [6]: Given a convex polygon P of n vertices, a set of $k \leq n$ vertex floodlights each of angle $\alpha = \pi/k$, placed one per vertex, can the lights always be oriented to fully illuminate *P*? It was proven in [6] that this question has negative answer for general k, but the question remains open for small values of k. (k = 4, k = 5).

It was shown in [6] that four $\pi/4$ vertex lights suffice to illuminate any quadrilateral. This result was improved in two ways: [5] has shown that four $\pi/4$ vertex lights are sufficient also for any convex polygon. In this paper, we prove that in the case of a quadrilateral, three $\pi/4$ vertex lights are actually sufficient.

[1] School of Information Technology and Engineering, Ottawa, Ontario, Canada
[2] Département D'informatique, Université du Québec à Hull, Hull, Québec, Canada

2 Illuminating a triangle

In this section, we study the problem of illuminating a triangle using three $\pi/6$ vertex floodlights.

Theorem : Every triangle can be illuminated by three $\pi/6$ vertex lights.

Sketch of the proof: The proof follows from the construction of Brocard, see [4]. Given a triangle ABC, draw a circle passing through vertex A and tangent at point B to segment BC. Repeat this construction two more times: once for vertex B, point C and segment CA and the second time for vertex C, point A and segment AC, see Fig.1. It may be proven that the three circles have a common point of intersection O inside the triangle. Moreover, it may be observed that the three angles $\angle$BAO, $\angle$CBO and $\angle$ACO are all equal to ω, such that:

$$\cot \omega = \cot A + \cot B + \cot C$$

Using this formula, it is possible to prove that $\omega \leq \pi/6$. Placing vertex floodlights of the angles $\angle$BAO, $\angle$CBO and $\angle$ACO we get an illumination using three equal floodlights of aperture at most equal to $\pi/6$.

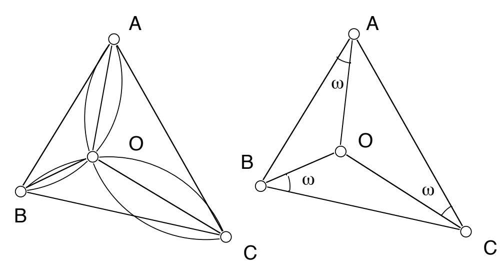

Fig.1 □

Is is obvious that for any given triangle, using it's Brocard point we get the smallest possible aperture for three vertex floodlights which mutually illuminate the triangle. As the two labelings of the triangle - the clockwise and the counterclockwise one - lead to two different Brocards points, we obtain then two different minimal floodlight illuminations. However, in both cases the Brocard angle is the same.

The equilateral triangle is the hardest one to illuminate using equal size floodlights, as in $\omega \leq \pi/6$, the equality holds only for the equilateral triangle ABC.

Another problem with floodlight illumination of a triangle is to find a minimal angle α, such that any three floodlights whose apertures sum up to α, will illuminate any triangle. We have the following:

Conjecture: Three vertex lights whose total angle is at least equal to $2\pi/3$ suffice to illuminate any triangle.

It is easy to see that $2\pi/3$ is also the lower bound for this problem.

3 Illuminating a Quadrilateral

In this section, we study the problem of finding an illumination for every quadrilateral with a minimum number of $\pi/4$ vertex lights. As mentionned before, it was shown in [6] that four $\pi/4$ vertex lights suffices to illuminate every quadrilateral. In this section, we are showing that actually three $\pi/4$ vertex lights are enough.

Theorem: Every quadrilateral can be illuminated by three $\pi/4$ vertex lights.

Sketch of the proof: We take into considerations the different cases of C, the smallest circle circumscribing quadrilateral P. We have:

Case1: $|C \cap P| = \{X, Y\}$, where X and Y are antipodal points on the boundary of C. Supposing that XY is a horizontal diameter of C, we consider two subcases:

1.1 XY is an edge for P. Two lights of $\pi/4$, placed at X and Y suffice to illuminate P.
1.2 XY is a diagonal of P. Then the proof differs depending on what are the quadrants of circle C containing the remaining two vertices of P. (see Fig.2):

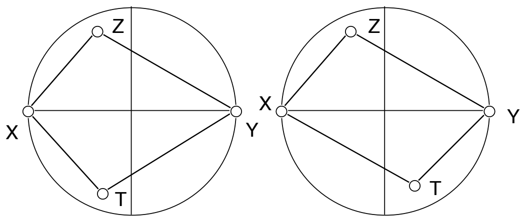

Fig.2

Case 2: $|C \cap P| = \{X, Y, Z\}$ where X , Y and Z are three vertices of P lying on the boundary of C. Without the loss of generality we assume that XY is the longest edge of the triangle XYZ. The proof differs depending on whether XY is an edge of P.

Case3: $|C \cap P| = \{X, Y, Z, T\}$. We assume that XY is the longest distance between all possible pairs of points of {X, Y, Z, T}. Again, the proof differs depending on whether XY is an edge of P.

□

The hardest quadrilateral to illuminate, using three equal size floodlights, is the regular one i.e. a square, proving that $\pi/4$ is the lower bound for this problem. In fact we think that any n-gon P may be illuminated by three floodlights, all of them sticking to the edges of P, and each one of the angle $(n-2)\pi/n$ - the half of the average angle of P.

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