Parallel edge flipping

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1 Introduction

Given a triangulation T of a set P of points on the plane, an edge e of T is *flippable* if it is incident to two triangles whose union is a convex quadrilateral C. By *flipping* e we mean the operation of removing e from T and replacing it by the other diagonal of C. In this way we obtain a new triangulation T' of P, and we say that T' has been obtained from T by means of a *flip*.

There are several reasons that make the study of flips in triangulations interesting. The first one is the existence of a simple greedy algorithm that constructs the Delaunay triangulation of a point set by succesive flips, starting from an arbitrary triangulation of the point set (see [4]). Another reason comes from the existence of a bijection between triangulations of a convex $(n+2)$-gon and binary trees with n internal nodes. Under this bijection, flipping an edge in a triangulation corresponds precisely to a *rotation* in the corresponding binary tree [9, 5]. Finally, let us mention that the flip operation also appears in other contexts, or applied to other kinds of triangulations [1, 2, 3, 7, 8, 10].

In a previous paper [6], the present authors studied several questions about flips in triangulations, mainly the question of how many flips are needed to transform a triangulation of a plane point set (or of a simple polygon) into another triangulation. Among other results, it was shown that two triangulations of a set of n points (or of a simple polygon with n vertices) in the plane are at most $O(n^2)$ flips apart. Moreover, pairs of triangulations were produced where $\Omega(n^2)$ flips are necessary, both for simple polygons and for point sets.

It seems natural to go one step further and to allow several edges to be flipped simultaneously, or *in parallel*. For this operation to make sense the edges must be independent, in the sense that no two of them can be sides of the same triangle. Let us call this operation a *parallel flip*. Then it is reasonable to expect that allowing parallel flips as a primitive operation the above $O(n^2)$ bound can be decreased substantially. The main purpose of this paper is to show that this is indeed the case.

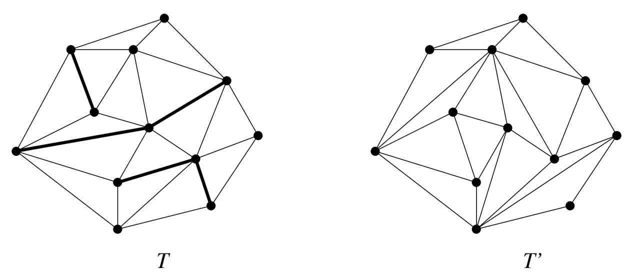

Figure 1: Flipping in parallel the set of thick edges of T produces T'.

We first prove that one can transform any triangulation of a convex n-gon into any other one with at most $O(\log n)$ parallel flips, and that this bound is tight. We remark that when translated in terms of binary trees, this result means that one can transform any binary tree into any other one using at most $O(\log n)$ "parallel rotations", a fact of independent interest.

For triangulations of general polygons and point sets we obtain an upper bound of $O(n \log n)$ parallel flips. We do not know whether this bound is tight, but we can show that $\Omega(n)$ is a lower bound for this problem.

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[*]Partially supported by Proyecto MEC-DGES-SEUID PB96-0005-C02-02 and Projecte UPC-PR9410. E-mail: hurtado@ma2.upc.es.

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Finally, we show that every triangulation of a set of n points contains a set of $(n-4)/6$ edges that can be flipped in parallel. In [6] we proved that every triangulation contains at least $(n-4)/2$ flippable edges.

2 Results

Our first result deals with convex polygons. Let us mention that, in the case of a convex n-gon, $O(n)$ ordinary flips always suffice to transform one triangulation into another (this is easy to prove, the hard problem is to determine the exact value for the maximum number of ordinary flips required [9]).

Theorem 1 *Any triangulation of a convex n-gon can be transformed into any other triangulation using at most $O(\log n)$ parallel flips, and this bound is tight.*

Besides its geometric content, this result can be translated into the language of binary trees, where parallel flips correspond to *parallel rotations*, that is, several rotations that take place without conflict simultaneously. While $\Omega(n)$ (sequential) rotations are eventually necessary to transform a binary tree into another one [9], Theorem 1 means that $O(\log n)$ parallel rotations are always sufficient.

Next we turn to triangulations of arbitrary polygons. We have already mentioned the example given in [6] of two triangulations of a polygon that are $\Omega(n^2)$ ordinary flips apart. Since a triangulation has a linear number of edges, this implies that the two triangulations are at least $\Omega(n)$ parallel flips apart. In order to obtain an upper bound the main ingredient is the following result, whose proof requires a series of technical lemmas.

Proposition 1 *Let T be a triangulation of a simple polygon Q_n and let e be a diagonal not in T. Then e can be introduced in T with at most $O(n \log n)$ parallel flips.*

The key fact with parallel flips is that the cost of introducing a single diagonal is the same as the cost of introducing all the diagonals of a triangulation, as shown in the following theorem.

Theorem 2 *Any triangulation T of a simple polygon Q_n can be transformed into any other triangulation T' using at most $O(n \log n)$ parallel flips.*

From this we can deduce the same result for triangulations of points sets. We remark that again the example of triangulations at (sequential) quadratic distance shows that $\Omega(n)$ is a lower bound.

Theorem 3 *Any triangulation T of a set P_n of n points on the plane can be transformed into any other triangulation T' using at most $O(n \log n)$ parallel flips.*

Finally, in the above mentioned paper [6], we proved that any triangulation of a set of n points contains at least $(n-4)/2$ flippable edges. Here we present an analogous result for parallel flips.

Theorem 4 *Every triangulation T of a set P_n of n points on the plane contains a set of at least $(n-4)/6$ edges of T that can be flipped in parallel. Also, for every n there exists a triangulation of a set of n points in which at most $(n-4)/5$ edges can be flipped in parallel.*

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