Efficient Regular Polygon Dissections

Evangelos Kranakis[13] Danny Krizanc[13] Jorge Urrutia[23]

[1] Carleton University, School of Computer Science, Ottawa, ON, K1S 5B6, Canada, {kranakis,krizanc}@scs.carleton.ca
[2] Instituto de Matemáticas, Universidad Nacional Autónoma de México
[3] Research supported in part by NSERC (Natural Sciences and Engineering Research Council of Canada) grants.

Abstract. We study the minimum number $g(m,n)$ (respectively, $p(m,n)$) of pieces needed to dissect a regular m-gon into a regular n-gon of the same area using glass-cuts (respectively, polygonal cuts). First we study regular polygon-square dissections and show that $\lceil n/2 \rceil - 2 \leq g(4,n) \leq \frac{n}{2} + o(n)$ and $\lceil n/4 \rceil \leq g(n,4) \leq \frac{n}{2} + o(n)$ hold for sufficiently large n. We also consider polygonal cuts, i.e., the minimum number $p(4,n)$ of pieces needed to dissect a square into a regular n-gon of the same area using polygonal cuts and show that $\lceil n/4 \rceil \leq p(4,n) \leq \frac{n}{2} + o(n)$, holds for sufficiently large n. We also consider regular polygon-polygon dissections and obtain similar bounds for $g(m,n)$ and $p(m,n)$.

Key Words and Phrases: Dissections, Glass-cuts, Polygonal cuts, Regular polygons, Squares.

1 Introduction

An old theorem from the first half of the nineteenth century by Lowry, Wallace, Bolyai and Gerwien asserts that any simple polygon can be dissected into a finite number of pieces and reassembled to form any other simple polygon of the same area (see [3]). It is easy to see that the dissection of one polygon into another depends on the shape of the polygons. In fact, for the dissection of a triangle to a square of the same area, the number of pieces depends on the length of the longest side of the triangle, e.g. see [3][page 222] and [1]. In particular, this raises the question of how the number of pieces in a dissection of one polygon into another depends on the number of vertices of the polygons. More specifically, we have the following interesting question:

Problem 1. What is the minimum number of pieces for dissecting a regular polygon into any other regular polygon of the same area?

A special case of this problem is when one of the regular polygons is of a simple type, such as a square.

Problem 2. What is the minimum number of pieces for dissecting a square into a regular polygon of the same area?

This second problem is simpler but also contains most of the essential observations in our study and will become the focus of attention of this paper. As a consequence of our analysis we will also obtain results concerning Problem 1.

It is interesting how many people (amateurs and mathematicians alike) have been occupied with this problem or its variations. Dissections are implicit in Hilbert's third problem: "Show that two tetrahedra having the same altitude and base area have the same volume without resort to the method of limits" [5].

Many attractive dissections have been the source of recreational activity and have been published in the entertainment sections of various publications. Of particular interest is the beautiful book [3] by G. Frederickson which also includes many such dissections.

1.1 Preliminaries

In the following we consider two types of dissections: *glass-cuts* and *polygonal cuts*. A glass-cut dissection cuts a simple polygon into two pieces using a straight line cut. A polygonal cut dissection cuts a simple polygon into two pieces along a polygonal line. Clearly, a glass-cut dissection is also a polygonal dissection, but not vice-versa as shown by the polygonal cut of the equilateral triangle depicted in Figure 9. A glass-cut dissection of an object A into another object B of the same area is *reversible* if there is a glass-cut dissection of B into A using exactly the same pieces. We note that all polygonal dissections are reversible, but this is not true for glass-cuts, as shown in Figure 9. By *piece* we understand a simple connected polygon whose interior is nonempty. The table in Figure 1 gives references for the best dissections of n-gons to a square of the same area for $n \leq 10$ and $n = 12$. Also note that for $n = 3, 5, 6$, the dissections are glass-cuts, while the remaining dissections are not.

n	# pieces	Author	page #	Glass-cut
3	4	Dudeney	137	yes
5	6	Brodie	120	yes
6	5	Busschop	118	yes
7	7	Theobald	129	no
8	5	Bennett	151	no
9	9	Theobald	132	no
10	7	Theobald	134	no
12	6	Lindgren	107	no

Fig. 1. Best-known dissection of a regular n-gon to a square of the same area with page references from Frederickson's book ([3]). The last column indicates whether or not the dissection is also a glass-cut.

It is interesting to note that all these dissections are derived from appropriate polygon tesselations by superimposing a tesselation of squares. As such, these techniques are not applicable to the study of our Problems 1 and 2. (See also [4].)

A fundamental result that will be used frequently in the remainder of this paper is Montucla's dissection (see Figure 2) for converting a rectangle R to any other rectangle R' of the same area (see also page 222 of [3]).

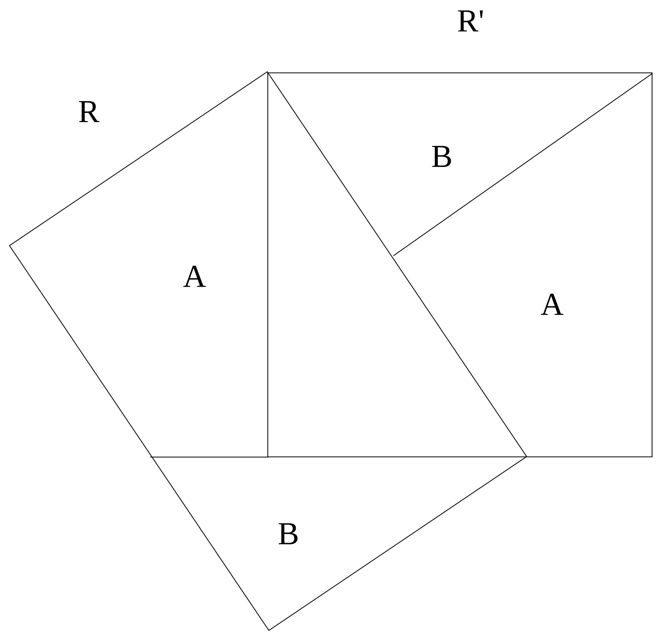

Fig. 2. Montucla's dissection of the rectangle R' of the same area as described in [3].

By width of a rectangle we understand the length of its shorter side. Given two rectangles of the same area and widths $w(R), w(R')$, respectively, the *width ratio* $w(R, R')$ is $\lceil w(R)/w(R') \rceil$, if $w(R') \leq w(R)$, and $\lceil w(R')/w(R) \rceil$, otherwise. Formally, we have the following result.

Theorem 1. *(Montucla, [3][page 222]) Let R, R' be two rectangles of the same area and width ratio $w(R, R')$. Then R can be dissected into R' with glass-cuts using at most $w(R, R') + 2$ pieces. In addition, if R is already dissected into p pieces, then by overlaying this dissection to the previous one we can dissect R' using at most $p(w(R, R') + 2)$ pieces. Moreover, if the original dissection of R is with glass-cuts, then so is the resulting dissection of R'.* □

It is interesting to see that when the width ratio of the given rectangles is constant, the number of pieces of the resulting dissection is a constant multiple of the number of pieces of the original dissection.

1.2 Results of the paper

A mathematical reformulation of the problems proposed above concerns the asymptotic behavior of the functions p, g which are defined as follows. For integers m, n, let $p(m,n)$ (respectively, $g(m,n)$) be the minimum number of pieces needed to dissect a regular m-gon into a regular n-gon using polygonal (respectively, glass-) cuts. We note that the following properties are immediate from the definitions

$$\begin{aligned} p(m,n) &= p(n,m), \\ p(m,n) &\leq g(m,n). \end{aligned}$$

In this paper we study these two functions in detail and prove the following theorems.

Theorem 2. *The following bounds hold for the functions p and g for all sufficiently large non-negative integers $m < n$:*

$$\begin{aligned} \max\{\lceil (n-m)/2 \rceil, \lceil n/3 \rceil\} &\leq g(m,n) \leq \tfrac{m}{2} + \tfrac{n}{2} + o(n), \\ \lceil (n-m)/4 + 1 \rceil &\leq p(m,n) \leq \tfrac{m}{2} + \tfrac{n}{2} + o(n), \end{aligned}$$

where $o(n)$ is a function of n such that $\lim_{n\to\infty} \frac{o(n)}{n} = 0$. □

Theorem 2 will be obtained as a corollary of the following theorem.

Theorem 3. *The following bounds hold for the functions p and g for all sufficiently large non-negative integers n:*

$$\begin{aligned} \lceil n/4 \rceil &\leq p(n,4) \leq \tfrac{n}{2} + o(n), \\ \lceil n/4 \rceil &\leq g(n,4) \leq \tfrac{n}{2} + o(n), \\ \lceil n/2 \rceil - 2 &\leq g(4,n) \leq \tfrac{n}{2} + o(n), \end{aligned}$$

where $o(n)$ is a function of n such that $\lim_{n\to\infty} \frac{o(n)}{n} = 0$. □

Details of the proofs as well as the dissection algorithms leading to the upper bounds will be given in Sections 2 and 3. The lower bounds are given in Section 4 and some open problems are proposed in Section 5.

2 Polygon-Square Dissections

This section provides new dissections of regular polygons into squares and estimates their asymptotic number of pieces. All the dissections below concern dissections of a regular n-gon into a square of the same area, which for simplicity is assumed to be equal to 1. The main theorem of this section is the following.

Theorem 4. *Let $k = k(n)$ be a function of n such that* $\lim_{n\to\infty} \frac{k(n)}{n} = 0$. *Any regular n-gon can be dissected to a square of the same area using at most $\frac{n}{2} + O(\frac{n}{k} \log k)$ glass-cuts. Conversely, this same dissection can be reversed to form a dissection of the square to a regular n-gon of the same area using only glass-cuts.*

The main ideas of the proof of this theorem are the following.

1. We dissect the regular n-gon into k-diamonds.
2. We dissect the k-diamonds into layers.
3. We assemble the layers into rectangles of varying widths.
4. We assemble the rectangles into a single rectangle.
5. We dissect the rectangle into a square.

In the next section we give the details of this construction.

2.1 Diamond dissections

The unit of dissection is a “circular” sector of the regular n-gon, called a k-*diamond*, and which is defined as follows. Pick an integer k. Consider the center of the regular n-gon (which is also the center of the circumscribing n-gon). Each sector is delimited by k adjacent sides of the n-gon and the two radii on the left-most and right-most vertex of this sequence of polygon sides. Depending on the parity of k there are two types of k-diamonds, shown in the two illustrations in Figure 3.

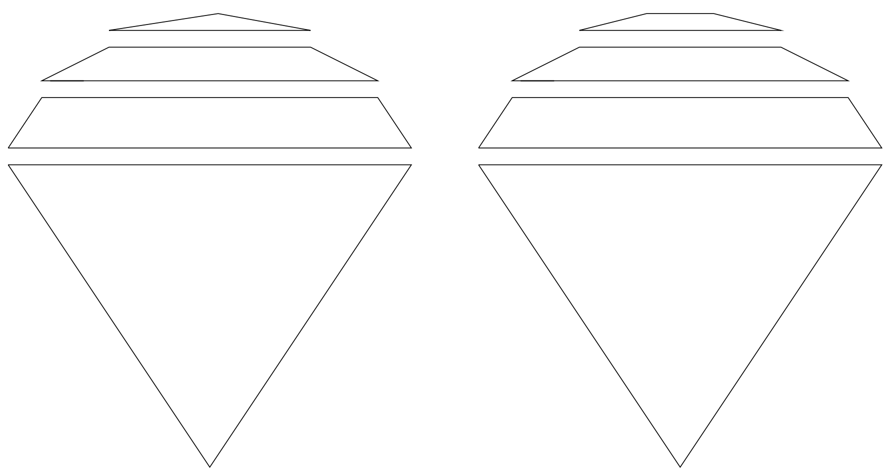

Fig. 3. A 6-diamond and a 7-diamond and their dissections into 4 layers.

Each sector is in turn dissected into $\lfloor k/2 \rfloor + 1$ layers.

For any vertex A of the regular n-gon, consider the diameter passing through A, and let A' be the point of intersection of this diameter with the perimeter of the n-gon. The first glass-cut is determined by the diameter from any arbitrary vertex, say A, of the n-gon. Observe that if n is even then A' is also a vertex of the n-gon, while if n is odd A' is the midpoint of a side of the n-gon, say S. In the case that n is odd we dissect the isosceles triangle delimited by S and the two equal radii adjacent to this side. If n is even then A' is also a vertex of the n-gon, in which case no extra dissection is needed.

We dissect the n-gon into $\lfloor n/k \rfloor$ non-overlapping sectors. If k does not divide n then a sector remains, consisting of $n - \lfloor n/k \rfloor k$ sides of the n-gon. It can be easily shown using Montucla's dissection, that this extra sector can be dissected into the "right" shape by adding only an extra overhead of $O(k)$ pieces. Details of the proof of this are left to the reader.

Without loss of generality we may assume that k is a fixed integer which is even and a divisor of n. First we dissect the n-gon into n/k k-diamonds. Then we dissect each k-diamond into $\frac{k}{2} + 1$ pieces as depicted in Figure 3. From top to bottom, the top layer is a triangle, the next $k/2 - 1$ layers are trapezoids and the last $(k/2+1)$-layer a triangle. It is clear that each layer consists of n/k pieces and therefore the total number of pieces is equal to $\frac{n}{2} + \frac{n}{k}$. We number the layers $0, 1, \ldots, \frac{k}{2}$ from top to bottom.

Straightforward calculations show that for a regular n-gon of area 1, the radius $a(n)$ of the circumscribing circle and side $b(n)$ of the n-gon are given by the following formulas:

$$a(n) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{2\pi/n}{\sin(2\pi/n)}}, \qquad b(n) = \frac{2}{n} \cdot \sqrt{\pi} \cdot \sqrt{\frac{2\pi/n}{\sin(2\pi/n)}} \cdot \frac{\sin(\pi/n)}{\pi/n}. \tag{1}$$

Also, $b(n)$ is the length of the base, and $a(n)$ the length of the equal sides of the 1-diamond. The height of the 1-diamond is equal to

$$h(n) = \frac{1}{\sqrt{\pi}} \cdot \frac{\pi/n}{\sin(\pi/n)} \cdot \sqrt{\frac{\sin(2\pi/n)}{2\pi/n}}. \tag{2}$$

Now we consider a trapezoid in the i-th layer and compute its dimensions. The height of the i-th trapezoid is equal to

$$h_i(n) = 2a(n)\sin(\pi/n)\sin((2i+1)\pi/n), \tag{3}$$

where $a(n)$ is the radius of the circle circumscribed in the regular n-gon and given in equation (1). Also the same formula gives the length $b(n)$ of the non-horizontal side of the trapezoid. Let $\ell_i(n)$ be the length of the longest base of the i-th trapezoid. Elementary calculations show that

$$\ell_i(n) = 4a(n)\sin(\pi/n)\sum_{j=0}^{i}\cos((2j+1)\pi/n). \tag{4}$$

It is also clear that the shortest side is equal to $\ell_{i-1}(n)$.

2.2 Assembling the layers into rectangles

Next we convert the trapezoids of the i-th layer into a rectangle. To accomplish this, we reflect half the trapezoids with respect to the x-axis and attach them two at a time as depicted in Figure 4 in order to form a parallelogram. Finally we cut a triangle from the leftmost end and attach it to the rightmost end of this parallelogram in order to form a rectangle.

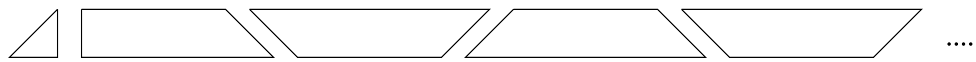

Fig. 4. The i-th layer of trapezoids. A right-angle triangle is dissected from the leftmost trapezoid and attached to the rightmost trapezoid.

The height of this rectangle is equal to $h_i(n)$ and its total length (if we let $s_i(n)$ be the length of the projection of the side of the i-th trapezoid on its large base) is equal to

$$\begin{aligned} L_i(n) &= \tfrac{n}{k}(\ell_i(n) - s_i(n)) \\ &= \tfrac{n}{k} 2a(n)\sin(\pi/n)\left(2\textstyle\sum_{j=0}^{i}\cos((2i+1)\pi/n) - \cos((2i+1)\pi/n)\right) \\ &= \tfrac{2\sqrt{\pi}}{k}(2i+1)A_i(n), \end{aligned} \tag{5}$$

where $A_i(n)$ is a function of n that converges to 1 as n goes to infinity and which is defined from the formula for $a(n)$, and by approximating the sum

$$2\sum_{j=0}^{i}\cos((2i+1)\pi/n) - \cos((2i+1)\pi/n)$$

by integrals (see [2][page 50]) and using the well-known limit formula

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

In addition, using equation (3) we observe that the sum of heights of these $k/2$ rectangles is equal to

$$\begin{aligned} \textstyle\sum_{i=0}^{k/2-1} h_i(n) &= 2a(n)\sin(\pi/n)\textstyle\sum_{i=0}^{k/2-1}\sin((2i+1)\pi/n) \\ &\leq 2a(n)\sin(\pi/n)\textstyle\sum_{i=0}^{k/2-1}\frac{(2i+1)\pi}{n} \\ &\approx \tfrac{\sqrt{\pi}}{n^2}k^2. \end{aligned} \tag{6}$$

At the same time we are interested in having the sum of the heights of these rectangles asymptotically small. In view of equation (6) the sum of the heights of these rectangles can be made arbitrarily small.

Each triangle of the $k/2+1$-layer is isosceles with equal sides of length $a(n)$, as given by Equation 1, and base of length $\ell_{k/2}(n)$, as given by Equation 4. Moreover its height is equal to $a(n)\cos(\pi n/k)$. Since by assumption $\lim_{n\to\infty}\frac{k(n)}{n} = 0$ it follows that the rectangle formed from the triangles of the $k/2+1$-layer has width approximately equal to $\sqrt{\pi}$ and height approximately equal to $1/\sqrt{\pi}$.

2.3 Forming a single rectangle

Now we have a sequence of rectangles $R_0, R_1, \ldots, R_{k/2}$ of varying lengths and heights which we will "normalize" to the same length. As indicated in equation (5), for $i = 0, \ldots, k/2-1$, R_i has height $h_i(n)$ and length $\frac{2\sqrt{\pi}}{k}(2i+1)$, while $R_{k/2}$ has dimensions approximately equal to $(1/\sqrt{\pi}) \times \sqrt{\pi}$. We now proceed to dissect the first $k/2$ rectangles to the length of rectangle $R_{k/2}$, which we denote by ℓ. Recall that ℓ is approximately equal to $\sqrt{\pi}$.

For each $i = 0, \ldots, k/2-1$, we dissect the rectangle R_i into $2(2i+1)$ subrectangles of the same height $h_i(n)$ and length approximately equal to $\frac{\ell}{k}$. The first $2(2i+1)-1$ of these subrectangles have length exactly equal to $\frac{\ell}{k}$. However, the last subrectangle will have length sufficiently close and approximately equal to $\frac{\ell}{k}$. We therefore use Montucla's dissection as in Theorem 1 to convert it to a subrectangle of length exactly $\frac{\ell}{k}$ at the cost of doubling the number of its trapezoidal pieces. The other subrectangles, however, remain unaffected. The total number of pieces of the i-th layer after this transformation is equal to

$$\frac{n}{k} + \frac{n}{2(2i+1)k}.$$

Adding these for $i = 0, 1, \ldots, k/2-1$, we obtain a total number of

$$\sum_{i=0}^{k/2-1} \left(\frac{n}{k} + \frac{n}{2(2i+1)k} \right) = \frac{n}{2} + \frac{n}{k} \sum_{i=0}^{k/2-1} \frac{1}{2(2i+1)} = \frac{n}{2} + O\left(\frac{n}{k} \log k \right) \quad (7)$$

pieces plus the n/k pieces of the bottom $k/2+1$-th layer.

Note that we have a total of

$$\sum_{i=0}^{k/2-1} 2(2i+1) = 2\left(\frac{k}{2} + 2 \sum_{i=0}^{k/2-1} i \right) = \frac{k^2}{2}$$

subrectangles each of length exactly $\frac{\ell}{k}$. Next we sort the subrectangles by height from the smallest to the largest and stack them up in order to form a rectangle of length $\sqrt{\pi}$. Let the subrectangles in sorted order from smallest to largest be $S_1, S_2, \ldots, S_{k^2/2}$. The stacking algorithm is as follows. Create a bucket B of length $\sqrt{\pi}$ consisting of k subbuckets $B_0, B_1, \ldots, B_k$ each of width $\sqrt{\pi}/k$ (see Figure 5).

Now place subrectangles $S_i, S_{i+k}, S_{i+2k}, \ldots, S_{i+(k/2-1)k}$ in subbucket B_i in this order from bottom-up, left-to-right, for $i = 1, 2, \ldots, k$.

The resulting subrectangles in the bucket do not yet form a rectangle. However, we claim that by using a total of at most k cuts on subrectangles and a total of at most $O(\frac{n}{k} \log k)$ new pieces we can convert it to a rectangle. To prove this we argue as follows. Let the heights of the corresponding subbuckets be $y_1, y_2, \ldots, y_k$. The height of the rectangle we are looking for must be equal to the average of these heights, i.e.,

$$a = \frac{y_1 + y_2 + \cdots + y_k}{k}.$$

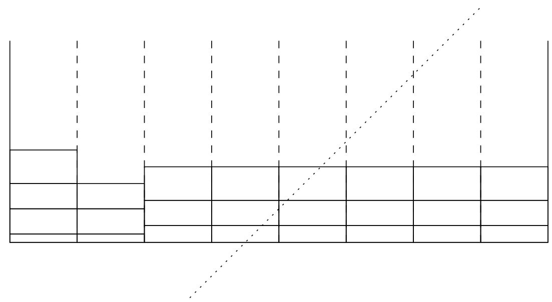

Fig. 5. The arrangements of subrectangles $S_0, S_1, \ldots, S_{k^2/2}$ in the subbuckets $B_1, B_2, \ldots, B_k$. The dotted line depicts a cut.

Let the height of the tallest subbucket be y_i. Dissect from the rectangle at the top of this subbucket a subrectangle of height exactly $y_i - a$. It is easy to see that there is a $j \neq i$ such that $y_j + y_i - a < a$. Therefore we can insert this subrectangle at the top of subbucket B_j. Moreover, subbucket B_i now has the required height, namely a. Now we remove the subbucket B_i and consider the remaining subbuckets

$$B_1, B_2, \ldots, B_{i-1}, B_{i+1}, \ldots, B_k.$$

It is also easily seen that the average of the heights of these remaining subbuckets is exactly a. Therefore our claim follows using induction on k. The total number of pieces thus added is easily shown to be at most $O(\frac{n}{k}\log k)$. Moreover, using the proof that led to equation (7) it can be shown that any cut starting from the top side of the rectangle and ending at the bottom side of this rectangle will intersect at most $O(\frac{n}{k}\log k)$ new pieces (see Figure 5).

2.4 Dissecting the rectangle into a square

We are now in a position to provide the final dissection. The construction of Section 2.3 gives rise to a rectangle consisting of two subrectangles; the rectangle of triangles and the rectangle of trapezoids (see Figure 6).

The two rectangles are dissected simultaneously to form a square using Montucla's dissection. This is depicted in Figure 7.

The argument of Section 2.3 shows that the extra overhead number of pieces is $O(\frac{n}{k}\log k)$. This completes the proof of Theorem 4. □

Proof of Theorem 3. Let ϵ be a positive real < 1. Let $c > 0$ be the constant of Theorem 4. Choose $k = n^{1-\epsilon}$. Then we define

$$o(n) = \frac{cn\log k}{k} = c(1-\epsilon)n^{\epsilon}\log n$$

and apply Theorem 4. This proves the upper bound stated in Theorem 3. □

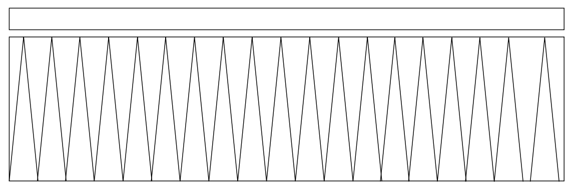

Fig. 6. The rectangle of triangles and trapezoids.

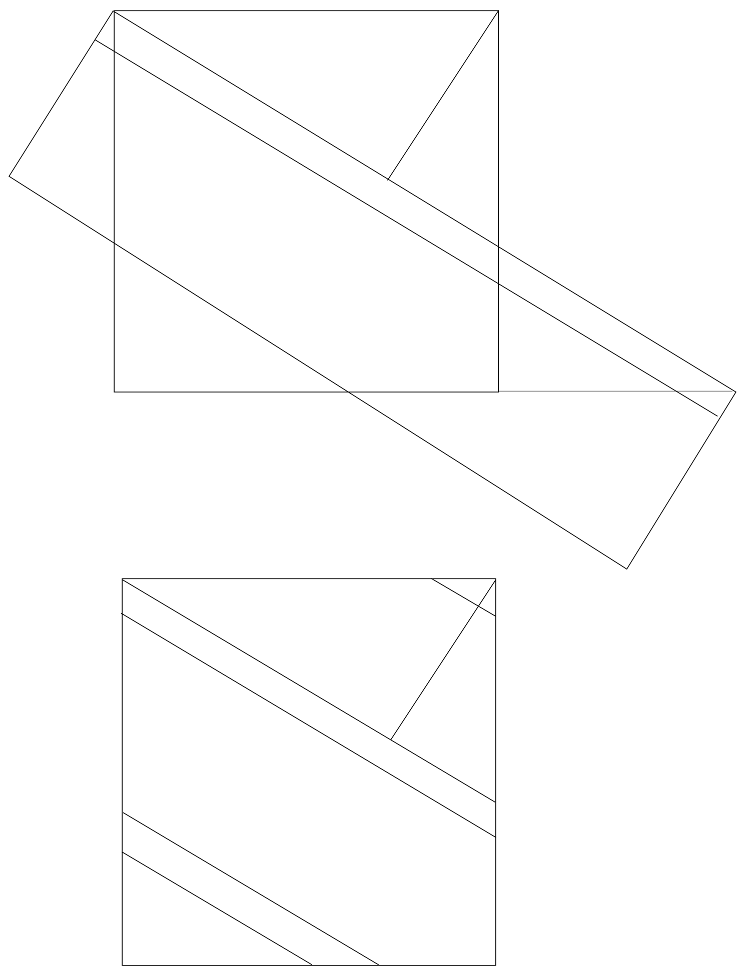

Fig. 7. Dissecting the two rectangles to a square. The leftmost picture depicts Montucla's dissection and the rightmost picture the result.

3 Polygon-Polygon Dissections

An immediate extension of the previous results is obtained through overlaying.

Theorem 5. *Let $k = k(m)$, $\ell = \ell(n)$ be functions of m, n, respectively, such that*

$$\lim_{m\to\infty} \frac{k(m)}{m} = \lim_{n\to\infty} \frac{\ell(n)}{n} = 0.$$

Any regular m-gon can be dissected to a regular n-gon of the same area using at most

$$\frac{m}{2} + \frac{n}{2} + O\left(\frac{mn}{k(m)\ell(n)}(\log k(m) + \log \ell(n))\right)$$

glass-cuts.

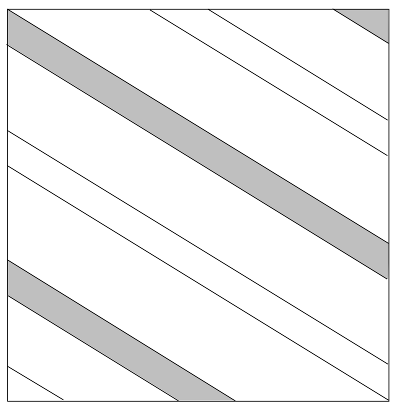

Fig. 8. We overlay the two squares arising from the dissections of the two regular polygons into squares. The rectangles of trapezoids are mapped into the thin strips and to distinguish them one is depicted with the shaded region.

Proof Consider the dissections of the m-gon and n-gon into a square. We rotate one of the two squares 180 degrees and overlay it over the other as depicted in Figure 8. We can estimate the total number of pieces. The m-gon (respectively, n-gon) dissection consists of $\frac{m}{k}$ (respectively, $\frac{n}{\ell}$) triangles and $\frac{m}{2}$ (respectively, $\frac{n}{2}$) trapezoidal pieces. Each of the $O(m/k)$ lines of the triangles intersects the trapezoidal pieces to generate a total of at most

$$\frac{m}{2} + O\left(\frac{m}{k}\frac{n}{\ell}\log \ell\right)$$

pieces. Similarly, each of the $O(n/\ell)$ lines of the triangles intersects the trapezoidal pieces to generate a total of at most

$$\frac{n}{2} + O\left(\frac{n}{\ell}\frac{m}{k}\log k\right)$$

pieces. Moreover, the intersection of the triangle lines generates a total of at most

$$O\left(\frac{m}{k}\frac{n}{\ell}\right)$$

pieces. Adding these estimates we obtain the desired result. □

The upper bounds stated in Theorem 2 are now obtained exactly as before. The lower bounds will be discussed in the next section.

4 Lower bounds

Note that glass-cuts are also polygonal cuts. Therefore lower bounds for polygonal cuts are also lower bounds for glass-cuts. An interesting observation is that a glass-cut dissection of a first polygon into a second is not necessarily a glass-cut dissection of the second polygon into the first (see Figure 9). This means that unlike the function $p(m,n)$, the function $g(m,n)$ is not necessarily symmetric in the variables m and n.

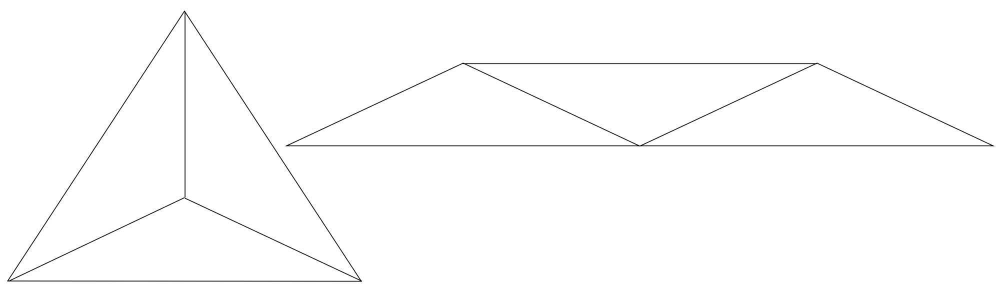

Fig. 9. An example of a dissection of an equilateral triangle into a quadrangle which is a glass-cut for the quadrangle but not a glass-cut for the triangle.

In this section we study lower bounds for polygonal cuts as well as glass-cuts. Section 4.1 is devoted to glass-cuts while Section 4.2 treats polygonal cuts.

4.1 Glass-Cuts

Theorem 6. *Any glass-cut dissection of a square into a convex n-gon of the same area requires at least $\lceil n/2 \rceil - 2$ pieces.*

Proof Consider a glass-cut dissection of the square into a convex n-gon consisting of k pieces. A glass-cut divides a given piece into two pieces, both of which are convex polygons. The sum of the angles of these two pieces exceeds the sum of the angles of the original piece by 2π, π or 0 depending on whether or not the two endpoints of the glass-cut segment intersect zero, one, or two vertices respectively, of the original convex piece. It follows that the sum of the angles of the pieces cannot exceed $2(k+1)\pi$. At the same time these k pieces can be reassembled to form the convex n-gon. Since the sum of the angles of the convex n-gon is exactly $(n-2)\pi$ it follows that

$$(n-2)\pi \leq 2(k+1)\pi.$$

This completes the proof of the theorem. □

Theorem 6 is valid for a glass-cut dissection of a square to a simple polygon and can also be generalized to give an $\lceil (n-m)/2 \rceil$ lower bound on the number of pieces for a glass-cut dissection of a convex m-gon to a convex n-gon, for $m < n$. If $n - m$ is small the lower bound is weak. For this reason we also give the following theorem.

Theorem 7. *Any glass-cut dissection of a regular m-gon into a regular n-gon of the same area, $m \neq n$, requires at least $\lceil n/3 \rceil$ pieces.*

Proof Consider a glass-cut dissection of the regular m-gon into a regular n-gon. Let the number of glass-cuts be equal to k. Each glass-cut dissects one convex piece into two and produces four new angles. The total number of pieces produced is $k+1$. Let V be the resulting set of vertices. Each of the n vertices of the n-gon must be among these vertices V. In addition, some of these n angles are composite (in the sense that it takes at least two of the angles produced by the dissections to form such an angle) and some are solid (i.e., not dissected by any line of a glass-cut). Let s be the number of solid angles. If we define by $w(v)$ the number of angles adjacent to vertex v then we have that $w(v) \geq 2$, for all vertices $v \in V$ which are not vertices corresponding to solid angles. The above discussion implies that

$$4(k+1) \geq \sum_{v \in V} w(v) \geq 2(n-s) + s.$$

This gives an $n/2 - s/4$ lower bound on the number of pieces.

Now we estimate the number s of solid angles. Let s_i and c_i be the number of solid and composite vertices produced by the i-th cut. Since $m \neq n$ it is clear that

$$s_i + c_i = 4, \text{ for all } i.$$

Moreover, since $m \neq n$ and the dissections are with glass-cuts, we have that $s_i \leq 2$, for all i. It follows that

$$s = \sum_{i=1}^{k} s_i \leq 2k.$$

Consequently,

$$k \geq \frac{n}{2} - \frac{s}{4} \geq \frac{n}{2} - \frac{k}{2},$$

which implies the desired lower bound and concludes the proof of Theorem 7. □

4.2 Polygonal Cuts

Both Theorems 6 and 7 are valid only for a dissection of a square to a regular n-gon. In the reverse direction, i.e., dissection of an n-gon to a square, we can only prove the more general (but weaker) lower bound of Theorem 8. Also note the importance of convexity of the given polygons. The lower bound is not valid if even one of the given polygons is not convex; it is easy to find a simple (non-convex) polygon which can be dissected into two pieces that can be assembled to form a square.

Theorem 8. *Any polygonal dissection of a convex n-gon into a square of the same area requires at least $\lceil n/4 \rceil$ pieces.*

Proof For any simple polygon let $c(P)$ and $r(P)$ be the number of convex and reflex vertices of P, respectively. Also let $\delta(P) = c(P) - r(P)$ be the convex-to-reflex vertex difference of the polygon P. The main observation is based on the following lemma.

Lemma 1. *If a polygonal cut dissects a simple polygon P into two simple polygon pieces Q and R then*

$$\delta(P) + 2 \leq \delta(Q) + \delta(R) \leq \delta(P) + 6.$$

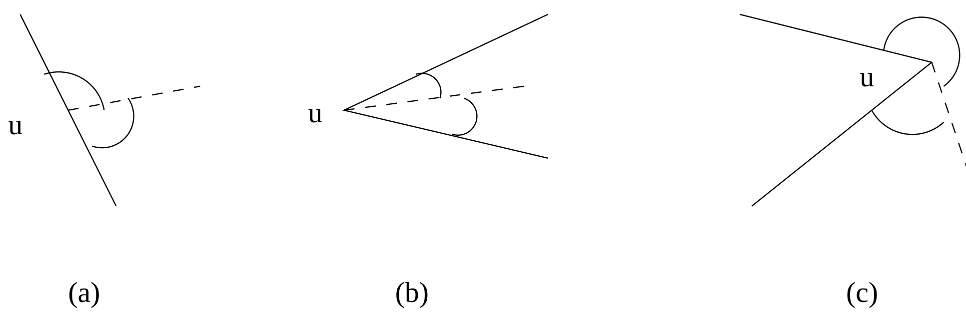

Fig. 10. Three types of cuts in a simple polygon through vertex u. In cases (a) and (b) two convex vertices are created, while in case (c) at most one reflex vertex may be created.

Proof of Lemma 1. To see this, observe that a polygonal cut at u creates either two convex vertices or one convex and one reflex vertex (see Figure 10).

However, for the vertices produced by the polygonal cut, with the possible exception of u, v, reflex vertices of Q match with convex vertices of R, and vice-versa; reflex vertices of R match with convex vertices of Q (see Figure 11).

Therefore a polygonal cut creates either the same number of new convex and reflex vertices or a surplus of either two or four new extra vertices. Hence, it is easy to derive the desired upper and lower bounds. This completes the proof of Lemma 1. □

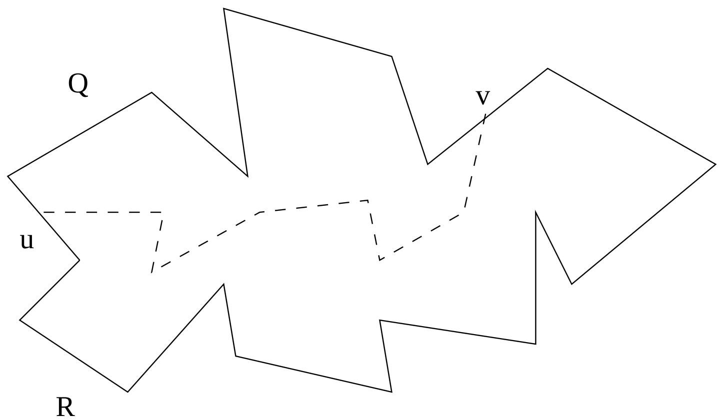

Fig. 11. Polygon P is dissected into two simple polygons Q and R with a polygonal line cut through u, v.

For convenience, we use the symbol $P \bowtie Q + R$ to denote that P is decomposed into pieces Q, R via a polygonal dissection. Now consider a polygonal dissection of the n-gon into k pieces $P_1, P_2, \ldots, P_k$. These pieces can be reassembled to form the square. Without loss of generality we may assume that the following sequence forms the square: define $P'_1 \bowtie P_1$ and by induction $P'_i = P'_{i-1} + P_i$, for $i \leq k$, where the last piece P'_k is the square. It follows by induction on $j \geq 0$, using Lemma 1, that

$$\sum_{i=1}^{k} \delta(P_i) \leq 6j + \delta(P'_{j+1}) + \sum_{i=j+2}^{k} \delta(P_i).$$

Indeed, assuming the inequality is valid for j, we can extend it to $j+1$ by using Lemma 1, i.e.,

$$\delta(P'_{j+1}) + \delta(P_{j+2}) \leq \delta(P'_{j+2}) + 4.$$

Since P'_k is the square, we have that $\delta(P'_k) = 4$. Hence for $j = k-1$ we obtain that

$$\sum_{i=1}^{k} \delta(P_i) \leq 6k.$$

At the same time, these same pieces can be reassembled to form the given convex n-gon. Without loss of generality we may assume that there is a permutation, say $Q_1, Q_2, \ldots, Q_k$, of the above pieces such that the following sequence forms the n-gon: define $Q'_1 \bowtie Q_1$ and by induction $Q'_i = Q'_{i-1} + Q_i$, for $i \leq k$, where the last piece Q'_k is the convex n-gon. The dissection must be such that at least the n convex vertices of the convex polygon are created. In particular, using Lemma 1 again, and arguing as before we obtain that

$$2k + n = 2k + \delta(Q'_k) \leq 2k + \sum_{i=1}^{k} \delta(Q_i) = 2k + \sum_{i=1}^{k} \delta(P_i) \leq 6k.$$

This concludes the proof of Theorem 8. □

We observe that the lower bounds stated in Theorem 3 are an immediate consequence of Theorems 7 and 8. The lower bounds of Theorem 2 are also obtained by adapting the proofs of Theorems 6, 7 and 8. Details are left to the reader.

5 Conclusions and Open Problems

We have investigated the problem of optimal dissections from a regular polygon to another polygon of the same area. We showed that asymptotically the optimal number of pieces in a glass-cut dissection of a square into a regular n-gon is exactly $\frac{n}{2} + o(n)$. Aside from tightening the previously obtained bounds it would be interesting to examine the answers to the following questions:

1. What is the asymptotic behavior of $g(n, 4)$, and more generally $g(m, n)$?
2. Are polygonal cuts more powerful than glass-cuts, i.e. is $g(m, n)$ asymptotically bigger than $p(m, n)$?
3. Is $g(m, n)$ asymptotically a symmetric function, i.e. is $|g(m, n) - g(n, m)| = o(m) + o(n)$?

It would also be interesting to look at dissections of other classes of polygons (e.g., star polygons).

References

1. M. J. Cohn, "Economical Triangle-Square Dissection", *Geometria Dedicata* **3**: 447–467, 1975.
2. T. H. Cormen, C. E. Leiserson, and R. L. Rivest, *Introduction to Algorithms*, MIT Press, 1990.
3. G. Frederickson, *Dissections Plane and Fancy*, Cambridge University Press, 1997.
4. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, New York, W. H. Freeman and Co., 1987.
5. D. Hilbert, Mathematishe Probleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1900. Subsequently in Bulletin AMS 8, pp 437–479, 1901–1902.