On the Chromatic Numbers of Some Flip Graphs

Ruy Fabila-Monroy† David Flores Peñaloza † Clemens Huemer ‡ Ferran Hurtado ‡
Jorge Urrutia † David R. Wood ‡

Abstract

In this paper we study the chromatic number of the following four flip graphs: A graph on the perfect matchings of the complete graph on $2n$ vertices and three graphs on the triangulations, Hamiltonian geometric non-crossing paths, and triangles respectively of a point set in convex position in the plane. We give tight bounds for the latter two cases and upper bounds for the first two.

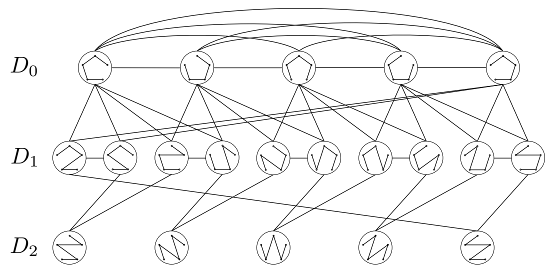

Figure 1: G_5

1 Introduction

Given a class $\mathcal{C}$ of combinatorial objects of a given kind and a transformation (flip) between these objects, a flip graph is defined as the graph whose vertex set is $\mathcal{C}$, where two vertices are adjacent whenever they differ by a flip. Flip graphs have received considerable attention in the past. Properties such as Hamiltonicity, connectivity and diameter have been widely studied [4, 6, 12, 15]. This interest is very likely due to the practical applications of these properties. For example, Hamiltonicity allows for rapid generation of the given combinatorial objects. We refer the interested reader to the survey [1].

The chromatic number $\chi(G)$ of a graph is the smallest integer such that it is possible to assign to each vertex of G an integer $i \leq \chi(G)$ such that adjacent vertices of G receive different integers. The chromatic number of flip graphs has received little attention, with only a few papers concentrating on this parameter (see for example [5]). In this paper we study the chromatic number of a flip graph on the perfect matchings of the complete graph and three flip graphs for sets of points in convex position, that is, they form the set of vertices of a convex polygon on the plane.

For flip graphs on convex point sets, we determine the exact chromatic number for geometric Hamiltonian paths (Section 2), we define a certain flip graph on its triangles of and determine its chromatic number up to a constant multiplicative factor (Section 3), and we give an upper bound on the chromatic number for triangulations (Section 5). We also consider a flip graph on the the perfect matchings of the complete graph on $2n$ vertices and give an upper bound on its chromatic number (Section 4). It should be stressed that in the case of matchings, no geometry is considered. We conclude with some open problems in Section 6.

Throughout the rest of the paper S will denote a set of n points in convex position in the plane.

2 Geometric Non-Crossing Hamiltonian Paths

Let G_n be the graph whose vertex set is the set of all non-crossing geometric paths with vertex set S. Two paths Γ_1 and Γ_2 in $V(G_n)$ are adjacent if and only if there exist edges e in Γ_1 and f in Γ_2 such that $\Gamma_2 = \Gamma_1 - e + f$. We say that Γ_2 is obtained from Γ_1 by *flipping* e and f. We point out that e and f may intersect. Note that Rivera-Campo and Urrutia-Galicia [14, 17] proved that G_n is Hamiltonian. We determine $\chi(G_n)$.

Theorem 1 *$\chi(G_n) = n$ for $n \geq 3$.*

Proof. Since $G_3 \simeq K_3$, assume $n \geq 4$. For $i = 0, \dots, n-3$, let D_i be the set of paths in $V(G_n)$ that contain exactly i non-convex hull edges (see Figure 1). Note that the set D_0 consists of n paths, all of whose edges are on the convex hull of S. Each element of D_0 is obtained by removing one edge of the convex hull of S. Clearly any two elements of D_0 are adjacent in

†Instituto de Matemáticas, Universidad Nacional Autónoma de México (ruy@ciencias.unam.mx, colegadavid@gmail.com, urrutia@math.unam.mx). Supported by CONACYT of Mexico, Proyecto SEP-2004-Co1-45876, and PAPIIT (UNAM), Proyecto IN110802.

‡Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain {(huemer.clemens,ferran.hurtado,david.wood}@upc.edu). Research of C. Huemer and F. Hurtado supported by projects MEC MTM2006-01267 and Gen. Cat. 2005SGR00692. Research of D. Wood supported by a Marie Curie Fellowship of the European Community under contract 023865.

G_n, and D_0 thus induces a complete subgraph of G_n. This proves that $\chi(G_n) \geq n$.

We now give a method to obtain an n-coloring of G_n. Note that each time we flip an edge of a path in D_i, we obtain a path in either D_{i-1}, D_i or D_{i+1}. Since D_0 induces a clique of size n in G_n, in any n-coloring of G_n we assign a different color to each element in D_0. We now show how to extend an n-coloring of the elements of D_0 to an n-coloring of G_n. Observe that every path Γ_1 in D_1 is adjacent to exactly two paths in D_0. Furthermore if Γ_1 and Γ_2 lie in D_1, then both are adjacent to the same two paths of D_0, or there is no path in D_0 adjacent to both of them. Since we are assuming $n \geq 4$, for each pair of adjacent paths in D_1 there are at least two colors we can assign to them, different from the colors assigned to their neighbors in D_0. Thus we can extend our coloring of D_0 to one of $D_0 \cup D_1$. Observe next that for for $i \geq 2$, there is no pair of adjacent paths in D_i, and that every path in D_i is adjacent to exactly two elements in D_{i-1}. Thus any n-coloring of $D_0 \cup \ldots \cup D_{i-1}$ can be extended to an n-coloring of $D_0 \cup \ldots \cup D_i$, $i = 2, \ldots, n-3$. Therefore $\chi(G_n) = n$. □

3 Triangle Graph

We introduce a new flip graph $G_\triangle(n)$, whose vertex set is the set of triangles with endpoints in S, two of which are adjacent if they share an edge and their interiors are disjoint. Assume that the elements of S are labeled $\{0, \ldots, n-1\}$ clockwise along the convex hull of S, and let $\triangle(i,j,k)$ denote the triangle with vertex set $\{i,j,k\}$.

Lemma 2 $\chi(G_\triangle(n)) \geq \log_2(n-1)$.

Proof. Let H be the subgraph of $G_\triangle(n)$ induced by all triangles containing 0 as a vertex. Suppose that H has a coloring with k colors. Let A_j be the set of colors assigned to the triangles $\triangle(0,i,j)$ with $i < j$. Observe that for $r \neq s$, $A_r \neq A_s$. This follows from the fact that if $r < s$, $\triangle(0,r,s)$ is adjacent to every triangle $\triangle(0,k,r)$ with $k < r$; thus the color assigned to $\triangle(0,r,s)$ is in A_s but not in A_r. The number of possible color sets must be at least $n-1$. Therefore $2^k \geq n-1$, $k \geq \log_2(n-1)$ and $\chi(G_\triangle(n)) \geq \log_2(n-1)$. □

The subgraph H of $G_\triangle$ considered in the proof of Lemma 2 is known as the *shift graph*, and its chromatic number is well known [16].

Given two graphs G and H, a homomorphism from G to H is a mapping from the vertex set of G to the vertex set of H such that adjacent vertices of G are mapped to adjacent vertices in H. It is well known and straightforward to see that if there is a homomorphism from G to H, then the chromatic number of G is less than or equal to the chromatic number of H.

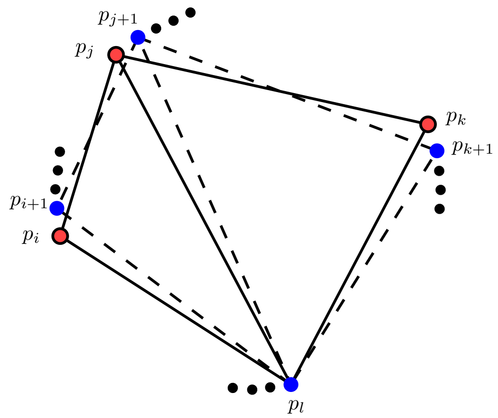

Figure 2: The homomorphism between G and $G_\triangle(m)$

Lemma 3 *Let G be the subgraph of $G_\triangle(2m)$ induced by all the triangles without edges on the convex hull of S. Then $\chi(G) \leq \chi(G_\triangle(m))$.*

Proof. Color the vertices of S red and blue such that no two consecutive vertices[1] have the same color. Note that the subgraph H of G induced by the blue points is isomorphic to $G_\triangle(m)$. We now map every triangle t in $V(G)$ to a triangle such that all its vertices are blue as follows: if i is a red vertex of t, substitute it by the blue vertex $i+1$, addition taken $\mod n$. Observe that triangles whose vertices were already all blue are mapped onto themselves. Since adjacent triangles in G are mapped to adjacent triangles in H, this mapping induces a homomorphism from G to H (see Figure 2). The result follows. □

Lemma 4 $\chi(G_\triangle(2m)) \leq \chi(G_\triangle(m)) + 3$.

Proof. We use a similar technique to that used in Theorem 1 of Section 2. Let G' be the subgraph of $G_\triangle(2m)$ induced by all those triangles of S that have at least one edge on the convex hull of S. Let $\tau = \triangle(i, i+1, j)$ be any such triangle. $S - \{j, i, i+1\}$ can be partitioned into two maximal subsets of consecutive points, namely the points from $i+1$ to j and the points from j to i, which we denote by l_τ and r_τ respectively. We define the "order" of τ to be $\min\{|l_\tau|, |r_\tau|\}$. For $i = 0, \ldots \lceil \frac{n-2}{2} \rceil$, let D_i be the subset of $V(G')$ of all triangles of order i. Note that the subgraph of G' induced by $D_{\lceil \frac{n-2}{2} \rceil}$ has maximum degree 2 and therefore chromatic number at most 3. Note that in general every vertex of D_i is only adjacent to at most 2 vertices in $D_{i+1} \cup \ldots D_{\lceil \frac{n-2}{2} \rceil}$. Thus any 3-coloring of $D_{i+1} \cup \ldots D_{\lceil \frac{n-2}{2} \rceil}$ can be extended to a coloring of $D_i \cup D_{i+1} \cup \ldots D_{\lceil \frac{n-2}{2} \rceil}$. Therefore $\chi(G') = 3$. By Lemma 3, we can color G with $\chi(G_\triangle(m))$ colors and G' with four different colors. This produces a coloring of $G_\triangle(2m)$ with $\chi(G_\triangle(m)) + 3$ colors. □

We now have:

Theorem 5 $\log_2(n-1) \leq \chi(G_\triangle(n)) \leq 3\lceil \log_2(n) \rceil - 6$.

Proof. Let $F(k) = \chi(G_\triangle(2^k))$. By Lemma 4, $F(k) \leq F(k-1) + 3$. Observe that $\chi(G_\triangle(4)) = 2$. Thus $F(k) \leq 3k - 6$ and therefore $\chi(G_\triangle(n)) \leq 3\log_2(n) - 6$.

For $n \neq 2^k$, let m be the smallest power of 2 greater than n. Since in general $G_\triangle(n)$ is a subgraph of $G_\triangle(n+1)$, we can color the vertices of $G_\triangle(n)$ with $\chi(G_\triangle(m)) \leq 3\log_2(m) - 6 = 3\lceil \log_2(n) \rceil - 6$ colors. □

4 Perfect Matchings of K_{2n}

Given any graph G, we define $\mathcal{M}(G)$ to be the graph whose vertex set is the set of perfect matchings of G, where two vertices of $\mathcal{M}(G)$ are adjacent whenever the symmetric difference of the corresponding perfect matchings is a cycle of length 4. $\mathcal{M}(G)$ is known as the *flip graph* of the perfect matchings of G.

We now give an upper bound on $\chi(\mathcal{M}(K_{2n}))$. To do so, we need the fact that $\chi(\mathcal{M}(K_{n,n})) = 2$. In [7], the flip graph of the non-crossing geometric matchings of a set of $2n$ points in convex position is shown to be bipartite. This graph is actually a subgraph of $\mathcal{M}(K_{n,n})$. Using the same arguments as in [7] it is possible to show that $\mathcal{M}(K_{n,n})$ is bipartite. For details, the interested reader can see [7].

Theorem 6 $\chi(\mathcal{M}(K_{2n})) \leq 4n - 4$ *for* $n \geq 2$.

Proof. Label the vertices of $V(K_{2n}) = \{1, \ldots, 2n\}$. For every perfect matching M of K_{2n}, let $U_M = \{i \in V(K_{2n}) | (i,j) \in M \text{ and } i > j\}$ and $D_M = \{i \in V(K_{2n}) | (i,j) \in M \text{ and } i < j\}$. Assign to every partition U_M, D_M of $V(K_{2n})$ given by a matching M the number $i_M = \sum_{i \in D_M} i \mod 2n-2$. Given two sets U, D, let $M_{U,D}$ be the set of matchings M such that $U_M = U$ and $D_M = D$. The subgraph $H_{U,D}$ of $\mathcal{M}(K_{2n})$ induced by $M_{U,D}$ is a subgraph of $\mathcal{M}(K_{n,n})$, and is thus 2-colorable. Color the vertices of H with colors i_M and i'_M.

We show now that if two matchings M and M' differ by a flip, they receive different colors.

Two cases arise: $U_M = U_{M'}$ and $D_M = D_{M'}$ or for some $i < j$, $U_M = U_{M'} - i + j$ and $D_M = D_{M'} - j + i$. In the first case, M and M' belong to H_{U_M, D_M} and thus receive different colors.

In the second case, $U_M \neq U_{M'} - i + j$, and the colors assigned to M and M' are different. Thus we obtain a coloring of the vertices of $\chi(\mathcal{M}(K_{2n}))$ with $2(2n-2) = 4n - 4$ colors. □

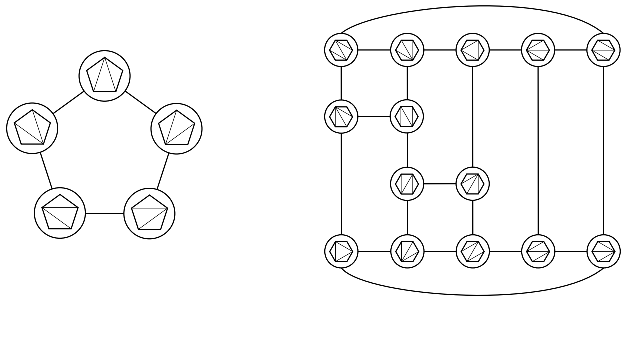

Figure 3: $G_T(5)$ and $G_T(6)$

5 Triangulations of a Convex Polygon

Finally, we consider a flip graph on the triangulations of S. Let $G_T(n)$ be the graph whose vertex set is the set of triangulations of S, where two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are adjacent in $G_T(n)$ whenever they differ by one edge flip; that is, there exist edges $e \in \mathcal{T}_1$ and $f \in \mathcal{T}_2$ such that $\mathcal{T}_2 = \mathcal{T}_1 - e + f$ (see Figure 3).

Much is known about $G_T(n)$. This is probably due to the fact that there is a bijection between triangulations of the n-gon and binary trees with $n-2$ nodes. A flip in a triangulation corresponds to a rotation in its corresponding binary tree.

Lucas [12] proved that $G_T(n)$ is Hamiltonian. Sleator et al. [15] proved that the diameter of $G_T(n)$ is $2n - 10$. Lee [10] proved that the automorphism group of $G_T(n)$ is the dihedral group of order $2n$ and can be realized as an $(n-3)$-dimensional convex polytope called the *associahedron*. Most of these results are proved again in [8] using a unifying framework called the *tree of triangulations*.

Of the flip graphs we have studied, $G_T(n)$ is the one for which we have made the least progress. We present our results as a starting point for further research in the area.

Theorem 7 $\chi(G_T(n)) \leq \lceil \frac{n}{2} \rceil$.

Proof. It is well known that for n even, the set of $\binom{n}{2}$ edges between the vertices of S can be partitioned into $\frac{n}{2}$ edge-disjoint geometric non-crossing graphs (see [11, 2] and Figure 4 for an example with $n = 6$). Since $G_T(n)$ is a subgraph of $G_T(n+1)$, we will assume n to be even for the time being. Label these graphs $G_1, \ldots, G_{\frac{n}{2}}$. If an edge e belongs to G_i, assign it the weight $w(e) = i$. To every triangulation $\mathcal{T}$ of S, assign the number $(\sum_{e \in \mathcal{T}} w(e)) \mod \frac{n}{2}$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two adjacent triangulations of $G_T(n)$, i.e. $\mathcal{T}_2 = \mathcal{T}_1 - e + f$ for some crossing edges e and f. Since e and f cross each other, $w(e) \neq w(f)$, and thus the numbers associated to $\mathcal{T}_1$ and $\mathcal{T}_2$ are different. This induces a coloring of $G_T(n)$ with $\frac{n}{2}$ colors for n even and $\lceil \frac{n}{2} \rceil$ in general. □

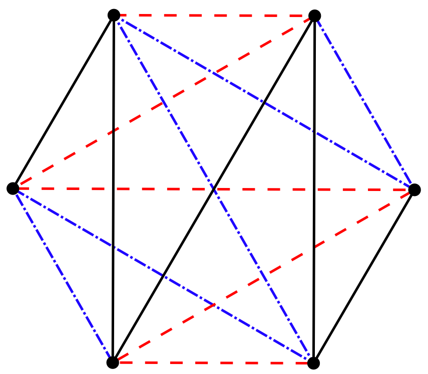

Figure 4: Partition of the edges into disjoint non-crossing geometric Hamiltonian paths

The upper bound on $\chi(G_T(n))$ in Theorem 7 is non-trivial since, for example, Brooks' Theorem [3] gives an upper bound of only $n-1$. However, $\chi(G_T(n))$ is in fact sublinear, as we now show. Johansson [9] proved that for sufficiently large Δ, every triangle-free graph with maximum degree Δ is $O(\frac{\Delta}{\log \Delta})$-colorable; see [13]. The next theorem follows since $G_T(n)$ is $(n-3)$-regular and triangle-free.

Theorem 8 $\chi(G_T(n)) \in O(\frac{n}{\log n})$.

We remark that the proof of Theorem 7 is constructive, whereas Johansson's proof is probabilistic.

6 Open Problems

The most challenging open problems arising from this work are to determine the chromatic numbers of $G_T(n)$ and $\mathcal{M}(K_{2n})$. Despite our efforts, we have not been able to obtain non-trivial lower bounds for these graphs. We have made the following educated guess of the chromatic number of $\mathcal{M}(K_{2n})$.

Conjecture 1 $\chi(\mathcal{M}(K_{2n})) = n+1$.

We have verified this conjecture with the aid of a computer for $n = 2, 3, 4$.

We have studied $\mathcal{M}(K_{2n})$, but it would be very interesting to study $\mathcal{M}(G)$ for other graphs.

There is another flip graph, $\mathcal{M}(G)$, related to $\mathcal{M}(G)$. $\mathcal{M}(G)$ has as its vertex set the perfect matchings of G, where two matchings are now adjacent if and only if their symmetric difference is a cycle of arbitrary length. We have yet to study its chromatic number.

With respect to $\chi(G_T(n))$, the problem seems far more intriguing since not only do we not have a non-trivial lower bound, we also believe that our upper bound is far from being tight.

Conjecture 2 $\chi(G_T(n)) = \Theta(\log(n))$.

It would be interesting to see what techniques will be capable of improving the lower bounds of both of these graphs. It is likely that such techniques will turn out to be useful in determining the chromatic number of other graphs.

References

[1] P. Bose and F. Hurtado. *Flips in planar graphs.* Technical Report TR-06-09, School of Computer Science, Carleton University, Ottawa, 2006.

[2] P. Bose, F. Hurtado, E. Rivera-Campo and D. R. Wood *Partitions of complete geometric graphs into plane trees.* Computational Geometry: Theory an Applications, 34(2):116-125,2006.

[3] R. L. Brooks. *On coloring the nodes of a network.* Cambridge Philos. Soc., 37:194–197, 1941.

[4] R. Cummins. *Hamilton circuits in tree graphs.* IEEE Transactions on Circuit Theory 13:82–90, 1966.

[5] V. Estivill-Castro, M. Noy and J. Urrutia. *On The chromatic number of tree graphs.* Discrete Math. 233:363–366, 2000.

[6] W. Goddard and H. C. Swart. *Distances between graphs under edge operations.* Discrete Math. 161:121–132, 1996.

[7] C. Hernando, F. Hurtado and M. Noy. *Graphs of non-crossing perfect matchings.* Graphs and Combinatorics, 18:517–532, 2002.

[8] F. Hurtado and M. Noy *Graph of triangulations of a convex polygon and tree of triangulations.* Computational Geometry: Theory and Applications, 13(3):179–188, 1999.

[9] A. Johansson. *Asymptotic choice number for triangle free graphs.* DIMACS Technical Report 91–95, 1996.

[10] C. Lee. *The associahedron and the triangulations of the n-gon.* European J. Combinatorics 10(4):551–560, 1989.

[11] D. E. Lucas. *Recreations Mathématiques*, Volume 2. Gauthiers Villars, Paris 1892.

[12] J. M. Lucas. *The rotation graph of binary trees is Hamiltonian.* J. Algorithms, 8(4):503–535, 1987.

[13] M. Molloy and B. Reed. *Graph colouring and the probabilistic method.*, Algorithms and Combinatorics, Vol. 23, Springer, 2002.

[14] E. Rivera-Campo and V. Urrutia-Galicia. *Hamilton cycles in the path graph of a set of points in convex position.* Computational Geometry: Theory and Applications, 18(2):65-72, 2001.

[15] D. Sleator, R. Tarjan and W. Thurston *Rotations distance, triangulations and hyperbolic geometry.* J. Amer. Math. Soc., 1:667–682, 1988.

[16] W. Trotter. *Ramsey theory and partially ordered sets.* Contemporary Trends in Discrete Mathmatics, R. L. Graham, et al., eds., DIMACS Series in Discrete Mathematics and Theoretical Computer Science 49:337–347, 1999.

[17] V. Urrutia-Galicia. *Algunas propiedades de gráficas geométricas.* Ph.D. thesis, Universidad Autónoma Metroploitana-Iztapalala, México D.F., 2000.