Maximal Number of Edges in Geometric Graphs without Convex Polygons

Chie Nara[1], Toshinori Sakai[1], and Jorge Urrutia[2]*

[1] Research Institute of Educational Development, Tokai University,
2-28-4 Tomigaya, Shibuya-ku, Tokyo 151-8677, Japan
{tsakai, cnara}@ried.tokai.ac.jp
[2] Instituto de Matemáticas, Ciudad Universitaria, Universidad Nacional Autónoma de México, México D.F., México
urrutia@matem.unam.mx

Abstract. A geometric graph G is a graph whose vertex set is a set P_n of n points on the plane in general position, and whose edges are straight line segments (which may cross) joining pairs of vertices of G. We say that G contains a convex r-gon if its vertex and edge sets contain, respectively, the vertices and edges of a convex polygon with r vertices. In this paper we study the following problem: Which is the largest number of edges that a geometric graph with n vertices may have in such a way that it does not contain a convex r-gon? We give sharp bounds for this problem. We also give some bounds for the following problem: Given a point set, how many edges can a geometric graph with vertex set P_n have such that it does not contain a convex r-gon?

A result of independent interest is also proved here, namely: Let P_n be a set of n points in general position. Then there are always three concurrent lines such that each of the six wedges defined by the lines contains exactly $\lfloor \frac{n}{6} \rfloor$ or $\lceil \frac{n}{6} \rceil$ elements of P_n.

1 Introduction

A geometric graph G is a graph whose vertex set is a set of P_n of n points on the plane in general position such that its edges are straight line segments joining some pairs of elements of P_n. Geometric graphs have received considerable attention lately, see for example a recent survey by J. Pach [6]. Some classical topics in Graph Theory have been studied for geometric graphs, e.g. Ramsey-type problems on geometric graphs have been studied in [4][5]. A classical problem in Graph Theory solved by Turán [7] is that of determining the largest number of edges that a graph has such that it does not contain a complete graph on r vertices. In this paper we study the corresponding problem for geometric graphs. We say that a geometric graph G *contains a convex r-gon* if its vertex and edge sets contain, respectively, the vertices and edges of a convex polygon with r vertices. We study the following problem: What is the largest

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number of edges a geometric graph may have in such a way that it does not contain a convex r-gon? In Section 2 we give tight bounds for our problem.

In Section 3 we study the following related problem: Given a point set P_n what is the largest number of edges that a geometric graph containing no convex r-gons may have such that its vertex set is P_n? Sharp bounds are given for the case $r = 5$ and $r = 7$. In this section we also prove the following problem which is interesting on its own right: Let P_n be a set of n points in general position. Then there are always three concurrent lines such that each of the six wedges defined by the lines contains exactly $\lfloor \frac{n}{6} \rfloor$ or $\lceil \frac{n}{6} \rceil$ elements of P_n. This is the discrete version of a theorem by R. C. and E.F. Buck [1] which states that any convex set can be divided by three concurrent lines into six parts of equal area.

2 Geometric graphs without convex r-gons

Let $T_{r-1}(n)$ be the Turán graph, that is the complete $(r-1)$-partite graph whose classes have sizes as equal as possible, and denote by $t_{r-1}(n)$ the number of edges in $T_{r-1}(n)$. We recall a result of Turán:

Theorem 1 *[7] The maximal number of edges in simple graphs of order n not containing a complete graph of order r is $t_{r-1}(n)$ and $T_{r-1}(n)$ is the unique graph of order n with $t_{r-1}(n)$ edges that does not contain a complete graph of r points.*

The next result follows immediately:

Theorem 2 *The maximum number of edges that a geometric r-graph whose vertices are in convex position is $t_{r-1}(n)$. The bound is tight.*

The following result of Erdös and Szekeres will be useful:

Theorem 3 *[2] Let k be a natural number. There exists a natural number $p(k)$ such that any set with at least $p(k)$ points on the plane, in general position, contains k points in convex position.*

It is well known that $p(4) = 5$, $p(5) = 9$, and that

$$2^{k-2} + 1 \leq p(k) \leq \binom{2k-5}{k-2} + 2$$

In fact it is conjectured that $p(k) = 2^{k-2} + 1$.

Our objective in this section is to prove the following result:

Theorem 4 *Let k be a natural number, $k \geq 3$, and let $r = p(k) - 1$. Then the maximal number of edges in a geometric graph on n points which does not contain a convex k-gon is $\lfloor \frac{r-1}{2r} n^2 \rfloor$. This bound is tight.*

Proof. Suppose that a geometric graph with n vertices contains more than $t_k(n)$ edges. Then by Turán's Theorem it contains a complete subgraph H with $p(k)$ vertices. By Theorem 3, the vertex set of H contains k elements in convex position, and thus G contains a convex k-gon.

To show that our bound is tight, take a point set S in general position with $p(k)-1$ points labeled $q_1, \ldots, q_{p(k)-1}$ on the plane such that S contains no k points in convex position, and let m be an integer greater than or equal to 0. If we substitute each of the points q_i by a set S_i with $r_i = m$, or $r_i = m+1$ points within a sufficiently small neighborhood of q_i, and join all pairs of points u, v with $u \in S_i$, $v \in S_j$, $i \neq j$, $i = 1, \ldots, p(k)-1$ we obtain a geometric graph with $n = r_1 + \ldots + r_{p(k)-1}$ vertices and $t_{p(k)-1}(n)$ edges which does not contain a convex k-gon. Our result follows. □

For the cases $r = 4$, and $r = 5$ it follows that any geometric graph containing no convex quadrilateral (respectively convex pentagons) contains at most $\lfloor \frac{3}{8} n^2 \rfloor$ (respectively $\lfloor \frac{7}{16} n^2 \rfloor$) edges.

Using the previous theorem we can construct geometric graphs containing no convex pentagons on n points as follows: Since $p(5) = 9$, take eight points such that no five of them are in convex position. Then substitute each point by m or $m+1$ points, $m \geq 1$, and finally join points that are in different subsets. The graph obtained contains $\lfloor \frac{7}{16} n^2 \rfloor$. See Figure 1. Similar constructions an be used to obtain geometric graphs without convex quadrilaterals.

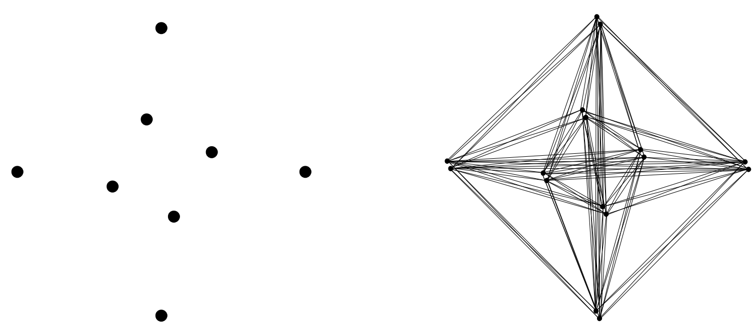

Fig. 1. Constructing a geometric graph with sixteen vertices, no convex pentagons, and $\lfloor \frac{7}{16} n^2 \rfloor$ edges, n=16.

3 Geometric graphs with predetermined vertex sets

To conclude our paper, we study the following problem: Let P_n be a point set. What is the largest number of edges that a geometric graph may have such that the vertex set of G is P_n, and it contains no convex r-gon?

We start by proving:

Theorem 5 *Let P_n be a point set. Then there is a geometric graph with vertex set P_n containing no convex pentagons with $\lfloor \frac{3}{8}n^2 \rfloor$. Our bound is tight.*

Proof. Assume w.l.o.g. that $P - n$ contains $4m$ points. It is well known that given any point set in general position with $4m$ points on the plane, there exist two intersecting lines such that there are exactly m points in the interior of each of the four wedges into which they divide the plane. Our geometric graph now contains the edges obtained by joining all pairs of points with elements in different wedges. See Figure 2. That the bound is tight follows from the fact that if the elements of P_n are in convex position, by Theorem 2 the graph cannot have more than $\lfloor \frac{3}{8}n^2 \rfloor$ edges. □

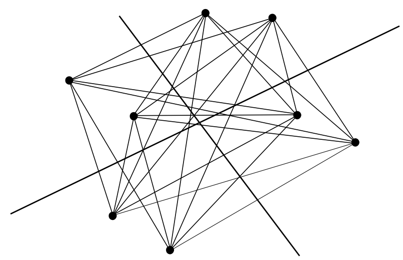

Fig. 2.

We finish by proving a similar result for geometric graphs containing no convex heptagons.

Theorem 6 *Let P_n be a point set. Then there is a geometric graph containing no convex heptagons with vertex set P_n with $\lfloor \frac{5n}{12} \rfloor$ edges. This bound is tight.*

To prove Theorem 6, we will use the following result which is interesting on its own:

Theorem 7 *Let P_n be a point set with n points in general position. Then there exist three concurrent lines, i.e. that intersect at a common point, such that the interior of each of the six wedges determined by them contains exactly $\lfloor \frac{n}{6} \rfloor$ or $\lceil \frac{n}{6} \rceil$ elements of P_n.*

Proof. We prove our result for $n = 6m$. Similar arguments can be applied for the remaining cases. Choose a horizontal line $\mathcal{L}_1$ that leaves $3m$ points in the interior of the semi-plane below it. Find a second line $\mathcal{L}_2$ with positive slope such that the four wedges $W_1, \ldots, W_4$ determined by them contain m, $2m$, m and $2m$ points respectively; see Figure 3(a). Let q be the point of intersection of our lines. Draw two rays emanating from q; the first one r_1, splitting W_2 into two wedges, each containing m points in their interiors. The second ray, r_2, splits W_4 in a similar way; see Figure 3(b).If r_1 and r_2 are collinear, we are done. Suppose

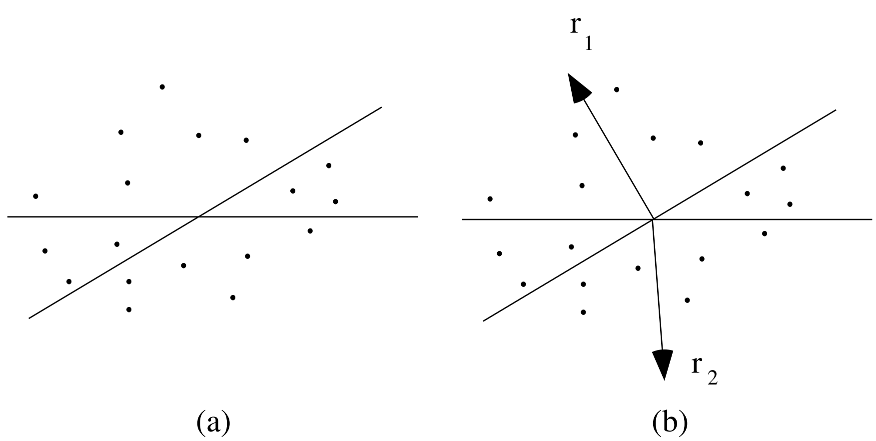

Fig. 3.

w.l.o.g. that the angle Θ_1 between r_1 and r_2 in the clockwise direction (the size of the angle we have to rotate r_1 to make it coincide with r_2) is greater than π.

Rotate $\mathcal{L}_1$ continuously keeping $3m$ points on each of the semi-planes determined by $\mathcal{L}_1$. Simultaneously update the second line, and r_1, and r_2. After a careful rotation of $\mathcal{L}_1$ 180 degrees, we can make $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide with themselves (but with different orientations), and r_1 (resp. r_2) fall on r_2 (resp. r_1). Along the way, Θ changed from its original size to $2\pi - \Theta$, and thus at some point it took the value π. At this point r_1 and r_2 became collinear. Our result follows. □

To prove Theorem 6, choose three lines as in the previous Theorem. As we did in Theorem 5, join all pars of points with elements in different wedges. Clearly the resulting geometric graph contains no convex heptagons. See Figure 4. To prove that our bound is tight, choose P_n in convex position.

At this point we were unable to prove a result similar Theorem 6 for convex quadrilaterals, hexagons. We ask the following question:

Problem 1 *Is it true that given a point set P_n of n points in general position, there is always a geometric graph containing no convex quadrilaterals (resp. hexagons) whose vertex set is P_n with $t_3(n)$ (resp. $t_5(n)$) edges?*

A similar question for convex r-gons $r \geq 8$ is open. However to solve this problems new techniques will be required, as Theorem 7 does not generalize to more than three lines. It is straightforward to find examples of point sets with $8m$ elements for which there are no four concurrent lines that split the point set into equal size subsets.

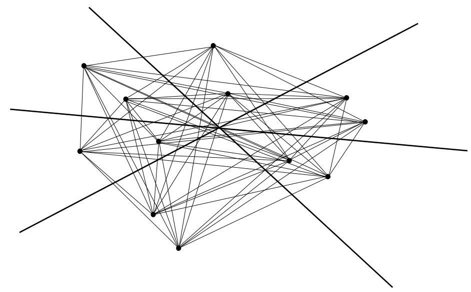

Fig. 4.

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