Hamiltonian Tetrahedralizations with Steiner Points

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Abstract

A tetrahedralization of a point set in 3-dimensional space is Hamiltonian if its dual graph has a Hamiltonian cycle. Let S be a set of n points in general position in 3-dimensional space. We prove that by adding to S at most $\lfloor \frac{m-2}{2} \rfloor$ Steiner points in the interior of the convex hull of S, we obtain a point set that admits a Hamiltonian tetrahedralization. We also obtain an $O(m^{\frac{3}{2}}) + O(n \log n)$ algorithm to solve this problem, where m is the number of elements of S on its convex hull. We also prove that point sets with at most 20 convex hull points have a Hamiltonian tetrahedralization without the addition of any Steiner points.

1 Introduction

Let S be a set of n points in $\mathbb{R}^3$ in general position. The convex hull of S ($Conv(S)$) is the intersection of all convex sets containing S.

The points of S lying on the boundary of $Conv(S)$ are called convex points and the points lying in the interior of $Conv(S)$, interior points.

A tetrahedralization $\mathcal{T}$ of S is a partition of $Conv(S)$ into tetrahedra with vertices in S such that:

1. The tetrahedra only intersect at points, lines or faces.
2. The tetrahedra do not contain points of S in their interior.

In a similar way, a triangulation of a point set in the plane is a partition of its convex hull into triangles satisfying the above properties.

Given a tetrahedralization $\mathcal{T}$ of S, we define $D_{\mathcal{T}}$, the dual graph of $\mathcal{T}$, to be the graph whose vertex set is the tetrahedra of $\mathcal{T}$, two of which are adjacent if and only if they share a common face.

In this paper, we are interested in tetrahedralizations such that their dual graph contains a Hamiltonian cycle or path. In general, we call such tetrahedralizations Hamiltonian tetrahedralizations. To differentiate between cycles and paths, we write Hamiltonian cycle and Hamiltonian path tetrahedralizations. We say that S admits a Hamiltonian tetrahedralization if there exists a Hamiltonian tetrahedralization of S.

A well known problem in computational geometry (see [5], Problem 29) asks if every convex polytope in $\mathbb{R}^3$ admits a Hamiltonian tetrahedralization, that is, a tetrahedralization of the set of vertices of the polytope.

The question was raised in [1], where a Hamiltonian triangulation was sought; in that paper the same problem was solved in the plane. In [3] it was proved that triangulations produced by applying Graham's Scan to calculate the convex hull of point sets are Hamiltonian.

It was observed in [1] that Hamiltonian triangulations allow for faster rendering of triangular meshes. The same holds true for tetrahedra. In [1], the problem of finding a Hamiltonian tetrahedralization for a convex polytope in $\mathbb{R}^3$ was conjectured to be NP-complete.

The existence of a Hamiltonian tetrahedralization of a convex polytope remains open. In this paper we study the following related problem: Given a convex polytope P in $\mathbb{R}^3$, how many Steiner points must be placed in the interior such that the set of vertices of P together with the added Steiner points admits a Hamiltonian tetrahedralization?

We consider the more general case and consider point sets rather than convex polytopes. Let S be a set of n points in $\mathbb{R}^3$ in general position such that its convex hull contains m vertices, and let m' be the number of S that belong to the interior of $Conv(S)$.

We present an algorithm that adds at most $\lfloor \frac{m-2}{2} \rfloor$ Steiner points, located in the interior of $Conv(S)$, to S. Our algorithm produces a Hamiltonian tetrahedralization. The overall complexity of the algorithm is $O(m^{\frac{3}{2}}) + O(n \log n)$.

Finally we show that if $m \leq 20$, no Steiner points need to be added.

2 The algorithm

The main idea is to first add a point to S to obtain a tetrahedralization such that its dual graph can be partitioned into cycles.

We then insert Steiner points to join existing cycles. We continue this process until the cycle partition consists of just one cycle. This final cycle is a Hamiltonian cycle in the dual graph of the final tetrahedralization.

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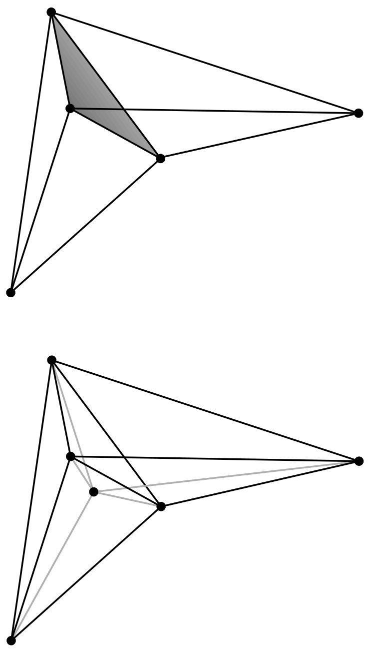

Figure 1: Join Operation.

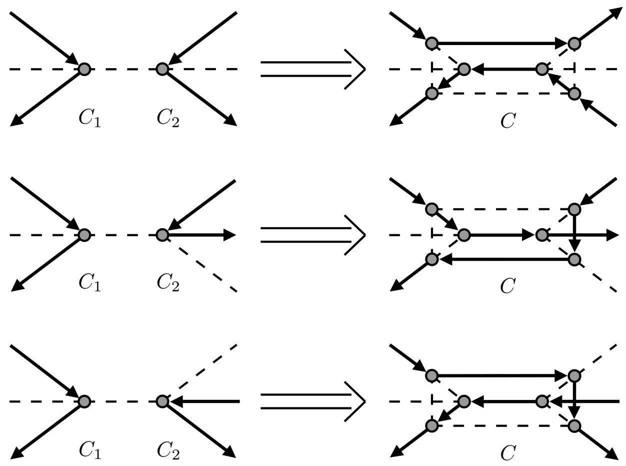

Figure 2: $D_{\mathcal{T}}$ before and after the join operation.

Actually, we first remove the interior points of S and those convex hull point sof degree 3 (that is, points adjacent to 3 other points in the boundary of $Conv(S)$). We can do this in view of the following:

Lemma 1 *If the convex hull points of S admit a Hamiltonian tetrahedralization, so does S.*

Proof. Consider an interior point x of S and suppose $S - \{x\}$ admits a Hamiltonian tetrahedralization $\mathcal{T}$. Let τ be the unique tetrahedron of $\mathcal{T}$ that contains x in its interior. If we remove τ from $\mathcal{T}$ and add the four tetrahedra induced by the faces of τ with x, we obtain a tetrahedralization of S and the Hamiltonian cycle of $D_{\mathcal{T}}$ can be extended to a Hamiltonian cycle of the new tetrahedralization. Applying this process recursively, the theorem follows. □

In the same manner we can suppose that S does not have any convex hull vertices of degree 3.

Theorem 2 *Let x be a convex hull point of S of degree 3. If $S - \{x\}$ admits a Hamiltonian tetrahedralization, then so does S.*

Proof. Suppose $S - \{x\}$ admits a Hamiltonian tetrahedralization $\mathcal{T}$. The three convex hull vertices of S adjacent to x form a face F of the boundary of $Conv(S - \{x\})$. Let τ_1 be the only tetrahedron of $\mathcal{T}$ that contains F as a face and let τ_2 be the tetrahedron induced by x and F. Clearly $\tau_1 \cup \tau_2$ is convex. If we remove τ_1 and τ_2 from $\mathcal{T}$ and replace them with the three tetrahedra induced by the faces of τ_1 (except F) and x, we obtain a tetrahedralization $\mathcal{T}'$ of S. The Hamiltonian cycle of $D_{\mathcal{T}}$ can now be extended to a Hamiltonian cycle of $D_{\mathcal{T}'}$

□

Assume now that S does not contain interior points or convex hull points of degree 3.

We insert a point p_0 in the interior of $Conv(S)$ and join every face of the boundary of $Conv(S)$ to it, forming a tetrahedralization $\mathcal{T}$ of $S \cup \{p_0\}$.

Let G be the graph induced by the 1-skeleton of the boundary of $Conv(S)$; that is, the graph whose vertex set consists of the convex hull points of S and whose edges are the edges of the boundary of $Conv(S)$. It is easy to see that both G and its dual graph are planar and 3-connected. By construction, the dual graph of G is isomorphic to $D_{\mathcal{T}}$. Since every face of G is a triangle, $D_{\mathcal{T}}$ is a regular graph of degree 3.

To obtain the initial partition, we use a theorem of Petersen [8] that states that every 2-connected cubic graph contains a perfect matching. Since $D_{\mathcal{T}}$ is 3-connected, in particular it is 2-connected and therefore contains a perfect matching M. If we remove the edges of M from $D_{\mathcal{T}}$, we obtain a regular graph of degree 2. This subgraph of $D_{\mathcal{T}}$ is the initial cycle partition.

2.1 Joining cycles

Consider two disjoint cycles, C_1 and C_2, in our cycle partition of $D_{\mathcal{T}}$, and furthermore suppose that there is an edge e of $D_{\mathcal{T}}$ that has its end points τ_1 and τ_2 in C_1 and C_2 respectively. Since τ_1 and τ_2 are tetrahedra in $\mathcal{T}$, e corresponds to a shared face F of τ_1 and τ_2.

The join operation consists of adding a point p to the interior of τ_1 so that the line segment joining the point q in τ_2 opposite to F in τ_2 intersects F. We now remove τ_1 and τ_2 and replace them by the six tetrahedra induced by the faces of τ_1, τ_2 and p (except F) as shown in Figure 1.

It can now be shown that there is a cycle that passes through all the vertices of $C_i \cup C_2 - \{\tau_1, \tau_2\}$ plus the six new tetrahedra containing p as a vertex (see Figure 2).

We repeat this process until a single cycle is obtained. We will show in the next section that the number of Steiner points we need to insert before a Hamiltonian cycle is reached is at most $\lfloor \frac{n-2}{2} \rfloor$.

3 Complexity and implementation.

In this section we will analyze the running time and implementation issues of the algorithm sketched in Section 2.

Suppose now that S is a point set with n points in $\mathbb{R}^3$ with m convex hull points and m' interior points, $m + m' = n$. We first calculate the convex hull of S in $O(n \log n)$, and then remove the points of S in the interior of $Conv(S)$.

Next, we remove the convex hull vertices of degree 3. This can be done in $O(m)$ by using a priority queue with all convex hull vertices of degree 3. Each time one is removed, the degree of its neighbors is checked and if necessary they are added to the queue.

Adding the first Steiner point p_0 and tetrahedralizing as in the previous section takes time $O(m)$.

The complexity of finding the initial cycle partition described at the end of Section 2 is that of finding a perfect matching in G. In a graph with $|V|$ vertices and $|E|$ edges, a perfect matching can be found in time $O(|E|\sqrt{|V|})$ [7]. Since we are dealing with a cubic graph, we have $|E| = \frac{3}{2}|V|$. Thus we can find the cycle cover in $0(\frac{3}{2}m\sqrt{m}) = O(m^{\frac{3}{2}})$ time.

Once we have the initial cycle cover, we return the vertices that were removed. This is done before the join operations in order to take advantage of the structure of the tetrahedralization to return the convex hull points of degree 3 and interior points efficiently. Using the fact that $D_{\mathcal{T}}$ is a planar graph, the interior points and convex hull points of degree 3 can be added using point location at a cost of $O(\log m)$ per point. The convex hull points of degree 3 are added first and the interior points afterwards. As these points are returned, the initial cycle partition is updated as in Lemma 1 and Theorem 2.

We have to be careful about the order in which the interior points are added. Suppose we have a tetrahedra τ which contains k interior points that remain to be added, and that we return q_0, one of these points. When we retetrahedralize the point set, τ would be split into 4 new tetrahedra. We have to guarantee that each of these tetrahedra receives a linear fraction of the points in τ, for otherwise the iterative process could take as much as $O(k^2)$. That is, we need a splitter vertex (see [2]). Such a vertex can be found in time $O(k)$, thus ensuring a total of $O(m' \log m')$ running time.

Finally we proceed to merge the set of cycles obtained thus far into a single cycle as in Subsection 2.1. Each time we join two cycles, we insert one Steiner point. Since G has m vertices, the number of faces of G is $2m - 4$, and since all the cycles obtained have at least four vertices, the initial cycle partition contains at most $\lfloor \frac{2m-4}{4} \rfloor$ elements. Thus the number of Steiner points required is at most $\lfloor \frac{m-2}{2} \rfloor$. This can be done in $O(n \log n)$ since there are $O(n)$ edges in H. The overall complexity of the algorithm is thus $O(m^{\frac{3}{2}}) + O(n \log n)$.

4 Hamiltonian Convex Hulls

To conclude the paper, we show that if the dual graph G defined by the convex hull of S is Hamiltonian, then no Steiner points need to be added.

Theorem 3 *Let S be a point set in $\mathbb{R}^3$ such that the dual graph H of G is Hamiltonian. Then S admits a Hamiltonian path tetrahedralization.*

Proof. Consider a planar embedding of H and a Hamiltonian cycle C of H. Let F be a face in this embedding such that all except one of its edges are in C.

Observe that there is a one-to-one mapping between the vertices of G and the faces of H. Let v be the vertex of G corresponding to F. Observe that each face of $Conv(S)$ (not containing v as one of its vertices) together with v induces a tetrahedron, and that the union of these tetrahedra forms a tetrahedralization of $Conv(S)$.

It is easy to see that the dual of this tetrahedralization is isomorphic to $H - F$, and thus contains a Hamiltonian path. □

Using Euler's formula and the fact that all 3-connected cubic planar graphs with 36 or fewer vertices have a Hamiltonian cycle (see [6]), we obtain the following corollary:

Corollary 4 *Let S be a point set in $\mathbb{R}^3$ having at most 20 convex hull points. Then S admits a Hamiltonian path tetrahedralization.*

The tetrahedralization mentioned in the proof of Theorem 3 (where all the points are joined to a given point) is known in the literature as a "pulling" tetrahedralization. Recently, point sets with no Hamiltonian path pulling tetrahedralizations have been shown to exist [4].

5 Conclusions

We presented an algorithm for computing Hamiltonian tetrahedralizations of a given point set S in $\mathbb{R}^3$ by adding Steiner points.

The algorithm has a running time of $O(m^{\frac{3}{2}}) + O(n \log n)$ and inserts at most $\lfloor \frac{m-2}{2} \rfloor$ Steiner points. We believe that this bound is not optimal.

We also showed that point sets with at most 20 convex hull points always admit a Hamiltonian path tetrahedralization.

We remark that have restricted ourselves to adding Steiner points to the interior of $Conv(S)$. If we allow the use of Steiner points in the exterior of $Conv(S)$, four exterior points (the vertices of a tetrahedron containing the elements of S in its interior) suffice.

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