Spanning trees of multicoloured point sets with few intersections

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Abstract

Kano et al. proved that if $P_0, P_1, \ldots, P_{k-1}$ are pairwise disjoint collections of points in general position, then there exist spanning trees $T_0, T_1, \ldots, T_{k-1}$, of $P_0, P_1, \ldots, P_{k-1}$, respectively, such that the edges of $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$ intersect in at most $(k-1)n - k(k-1)/2$ points. In this paper we show that this result is asymptotically tight within a factor of $3/2$. To prove this, we consider *alternating* collections, that is, collections such that the points in $P := P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ are in convex position, and the points of the P_i's alternate in the convex hull of P.

1 Introduction

Throughout this paper we consider collections $\{P_0, P_1, \ldots, P_{k-1}\}$ of point sets in the plane. Our interest lies in the following question: what is the minimum number of intersections among the edges of a collection $\{T_0, T_1, \ldots, T_{k-1}\}$ of spanning trees for $\{P_0, P_1, \ldots, P_{k-1}\}$, respectively?

In order to avoid unnecessary complications, it makes sense to assume that our collections P_i satisfy certain properties. It is pointless to consider the case in which some P_i are empty. Similarly, having two different P_i's with nonempty intersection, or having that $\cup_{i=0}^{k-1} P_i$ is not in general position leads to pathological situations. With these observations in mind, we arrive to the following definition.

Definition *A collection of $\{P_0, P_1, \ldots, P_{k-1}\}$ of point sets in the plane is* good *if (i) each P_i is nonempty; (ii) the P_i's are pairwise disjoint; and (iii) $\cup_{i=0}^{k-1} P_i$ is in general position.*

Let $\{P_0, P_1, \ldots, P_{k-1}\}$ be a good collection of point sets in the plane. A *corresponding set of trees for $\mathcal{P}$* is a collection $\mathcal{T} = \{T_0, T_1, \ldots, T_{k-1}\}$ such that T_i is a spanning tree for P_i, for $i = 0, \ldots, k-1$.

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Given a set of trees $\{T_0, T_1, \ldots, T_{k-1}\}$, its *intersection number* $\mathrm{int}(\{T_0, T_1, \ldots, T_{k-1}\})$ is the total number of pairwise intersections of edges in $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$.

With this terminology, our problem of interest outlined above can be paraphrased as follows.

Question *Let $\{P_0, P_1, \ldots, P_{k-1}\}$ be a good collection of point sets in the plane. What is the minimum $\mathrm{int}(\{T_0, T_1, \ldots, T_{k-1}\})$ taken over all corresponding sets of trees $\{T_0, T_1, \ldots, T_{k-1}\}$ for $\{P_0, P_1, \ldots, P_{k-1}\}$?*

This question was fully answered for the case $k = 2$ by Tokunaga [2]. However, the methods developed by Tokunaga do not seem to extend to $k > 2$.

In [1], Kano et al. gave the following general upper bound.

Theorem 1 (Kano et al.) *Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be a good collection of point sets in the plane, and let $n = |\cup_{i=0}^{k-1} P_i|$. Then there is a corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$ such that $\mathrm{int}(\mathcal{T}) \leq (k-1)n - k(k-1)/2$.*

Naturally, such a bound is of interest only if it is not too far from optimal. In the same paper, they also proved that, indeed, this bound is asymptotically tight up to a constant factor.

Theorem 2 (Kano et al.) *For each fixed k, the bound of Theorem 1 is asymptotically within a factor of 2 from the optimal solution.*

One of the highlights and main motivations of the present work is a proof that the bound in Theorem 1 is even tighter than as established in Theorem 2.

Theorem 3 *For each fixed k, the bound of Theorem 1 is asymptotically within a factor of $3/2$ from the optimal solution.*

This improvement is actually a straightforward consequence of an exhaustive analysis we perform on the special case in which the points in $\cup_{i=0}^{k-1} P_i$ are in convex position, and satisfy certain alternation condition (see proof after Theorem 10).

To test the tightness of Theorem 1, one needs to look for collections $\mathcal{P}$ for which the edges in *any* corresponding set of trees intersect a large number of times.

While seeking for such collections $\mathcal{P}$, it is quite natural to explore collections for which the points in $\cup_{i=0}^{k-1} P_i$ are in convex position. Moreover, it is intuitively appealing to propose that the points in $\cup_{i=0}^{k-1} P_i$ be arranged so that the points of each P_i "alternate as much as possible" with the points of the other P_j's.

Definition *A good collection $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ is alternating if the points in $\cup_{i=0}^{k-1} P_i$ are in convex position, and they can be labelled $p_0, p_1, \ldots, p_{sk-1}$, so that they appear in this cyclic order in the convex hull of $\cup_{i=0}^{k-1} P_i$ and, moreover, $P_i = \{p_i, p_{i+k}, \ldots, p_{i+(s-1)k}\}$, for $i = 0, 1, \ldots, k-1$.*

Note that, in particular, if $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ is alternating then $|P_0| = |P_1| = \cdots = |P_{k-1}|$.

In [1], Kano et al. considered alternating collections, and described constructions of corresponding sets of trees for every $k \geq 3$. These constructions yield the following.

Proposition 4 (Kano et al. [1]) *Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be an alternating collection, where $k \geq 3$ and $n := |\cup_{i=0}^{k-1} P_i| \geq 2k$. Then there is a corresponding set of trees $\mathcal{T}_c$ for $\mathcal{P}$ such that $\mathrm{int}(\mathcal{T}_c) = (3k^2/4 - k)((n/k) - 1) - k(k-2)/4$ if k is even, and $\mathrm{int}(\mathcal{T}_c) = (3(k-1)^2/4 + (k-1)/2)((n/k) - 1) - (k-1)^2/4$ if k is odd.*

The constructions behind Proposition 4 are so natural that Kano et al. conjectured that they are best possible.

Conjecture 5 (Kano et al. [1]) *Suppose that $\mathcal{P}$ satisfies the hypotheses of Proposition 4, and let $\mathcal{T}_c$ be the corresponding set of trees for $\mathcal{P}$ given by Proposition 4. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, $\mathrm{int}(\mathcal{T}) \geq \mathrm{int}(\mathcal{T}_c)$.*

One of the central results in this paper is the proof of Conjecture 5 for $k = 3$ (and also for $k = 4$; see Theorems 8 and 9). For $k = 3$, Proposition 4 claims that if $\mathcal{P} = \{P_0, P_1, P_2\}$ is alternating, and $n := |P_0 \cup P_1 \cup P_2| \geq 6$, then there is a corresponding set of trees $\mathcal{T}_c$ such that $\mathrm{int}(\mathcal{T}_c) = (4/3)n - 5$. Thus the following statement settles Conjecture 5 for $k = 3$. The proof is in Section 2.

Theorem 6 *Let $\mathcal{P} = \{P_0, P_1, P_2\}$ be an alternating collection such that $n := |P_0 \cup P_1 \cup P_2| \geq 6$. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, $\mathrm{int}(\mathcal{T}) \geq (4/3)n - 5$. Thus Conjecture 5 holds for $k = 3$.*

Keeping in mind that the motivation behind Conjecture 5 was to search for collections $\mathcal{P}$ for which any corresponding set of trees has large intersection number, it is natural to ask if dropping the condition that $\mathcal{P}$ is alternating can yield still better (or at least comparable) results.

That is, is there a (non alternating) collection $\mathcal{P} = \{P_0, P_1, P_2\}$ for which *every* corresponding set of trees $\mathcal{T}$ has $\mathrm{int}(\mathcal{T}) \geq (4/3)|P_0 \cup P_1 \cup P_2| - 5$?

We pursued this question, and came out with a definite answer (for the proof see Section 3).

Theorem 7 *Let $\mathcal{P} = \{P_0, P_1, P_2\}$ be a non alternating good collection such that $n := |P_0 \cup P_1 \cup P_2| \geq 6$. Then there is a corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$ such that $\mathrm{int}(\mathcal{T}) < (4/3)n - 5$.*

Thus, if our interest lies (as it happens) in collections $\mathcal{P}$ such that $\mathrm{int}(\mathcal{T})$ is large for *every* corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, then our best bet is to focus on alternating collections.

What about alternating collections with $k > 3$? Our first result in this regard is the following general statement, whose proof is in Section 4.

Theorem 8 *If Conjecture 5 holds for some k odd, then it also holds for $k+1$.*

In combination with Theorem 6, this immediately yields the following.

Theorem 9 *Let $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ be an alternating collection such that $n := |P_0 \cup P_1 \cup P_2 \cup P_3| \geq 8$. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, $\operatorname{int}(\mathcal{T}) \geq 2n - 10$. Thus Conjecture 5 holds for $k = 4$.* ∎

Theorem 6 also yields a nontrivial bound for alternating collections for all other values of k.

Theorem 10 *Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be an alternating collection such that $k \geq 3$ and $n := |\cup_{i=0}^{k-1} P_i| \geq 2k$. Then, for any corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$, $\operatorname{int}(\mathcal{T}) \geq (2/3)\big[(k-1)n\big] - 5k(k-1)/6$.*

This last statement follows from a standard counting argument from the case $k = 3$. The proof is in Section 5.

We conclude this introductory section with the observation that Theorem 10 implies the tightness of Theorem 1 we claimed in Theorem 3.

Proof of Theorem 3. It follows immediately from Theorem 10. ∎

2 Alternating three–coloured collections: proof of Theorem 6

Since $\mathcal{P}$ is alternating, we may assume that the points in $P = P_0 \cup P_1 \cup P_2$ are labelled so that $P_0 = \{p_0, p_3, \ldots, p_{3s-3}\}, P_1 = \{p_1, p_4, \ldots, p_{3s-2}\}$, and $P_2 = \{p_2, p_5, \ldots, p_{3s-1}\}$, in such a way that the points appear in the convex hull of P in the cyclic order $p_0, p_1, p_2, \ldots, p_{3s-1}$. Note that $n \geq 6$ implies $s \geq 2$.

We proceed by induction on s. The proof for $s = 2$ is straightforward.

Thus, we assume that the statement is true for $s = t - 1$, where $t \geq 3$, and consider the case $s = t$.

Let $\{T_0, T_1, T_2\}$ be a corresponding set of trees for $\{P_0, P_1, P_2\}$. Our aim is to show that the edges in $T_0 \cup T_1 \cup T_2$ intersect at least $(4/3)(3s) - 5 = 4s = 5$ times.

A vertex in P_i is an *i–vertex*. An edge in T_i is an *i–edge*.

A *crossing* is an intersection of edges in $T_0 \cup T_1 \cup T_2$.

Note that if every edge in $T_0 \cup T_1 \cup T_2$ has at least 3 crossings, then the total number of intersections in $T_0 \cup T_1 \cup T_2$ is at least $3(3s)/2 > 4s - 5$. By relabelling the points in $\mathcal{P}$ if necessary (perhaps even reversing the cyclic order of the points in $\mathcal{P}$), we may assume that some 0–edge e_0 has at most 2 crossings, and, moreover, that the vertices incident with e_0 are p_0 and p_{j_0}, with $j_0 \geq 6$.

It is readily checked that connectivity considerations (of the trees T_i) imply that every 0–edge intersects at least one 1–edge and at least one 2–edge. We therefore conclude that one of the crossings of e_0 occurs with a 1–edge e_1, and the other one with a 2–edge e_2. Let p_{i_1}, p_{j_1} (respectively p_{i_2}, p_{j_2}) denote the end vertices of e_1, labelled so that $0 < i_1, i_2 < j_0$ and $j_0 < j_1, j_2 < 3t - 1$.

A crossing is *internal* if both edges involved in it belong to $\{e_0, e_1, e_2\}$. A crossing is *external* if it is not internal and it involves an edge in $\{e_0, e_1, e_2\}$. A crossing is *good* if it is either internal or external.

The following statement shows that, in order to take care of the inductive step, it suffices to prove that e_0, e_1, e_2 are involved in a sufficiently large number of crossings.

Claim 11 *In order to deal with the inductive step, it suffices to show that at least one of the following conditions holds.*

(i) e_0, e_1, e_2 *are incident with leaf vertices that appear consecutively in* $\mathcal{P}$*, and there are at least* 4 *good crossings.*

(ii) There are at least 5 *good crossings.*

(iii) There are at least 4 *good crossings, and* e_1, e_2 *are both incident with leaf vertices.*

Proof. Suppose that (i) holds. Remove e_0, e_1, and e_2, and the consecutive leaves in $\mathcal{P}$ that are incident with these edges. This removes at least 4 crossings, by assumption. The result is a collection with $3(s-1)$ points to which the inductive hypothesis can be applied, to obtain at least $4(s-1) - 5 = 4s - 9$ crossings. These $4s - 9$ crossings, together with the 4 crossings previously removed, yield at least $4s - 5$ crossings in $T_0 \cup T_1 \cup T_2$, thus completing the inductive step.

Suppose now that (ii) holds. Suppose first that neither e_1 nor e_2 is incident with a leaf vertex. Remove e_1 and e_2. This removes at least 5 crossings, since e_0 by assumption only crosses e_1 and e_2. If we now contract e_0 (along with all its incident edges), collapsing p_0 and p_{j_0} and replacing them by a vertex placed in any point of e_0, we obtain two separate nonempty collections of points, of sizes $3s'$ and $3s''$, with $s' + s'' = s$, to which the inductive hypothesis can be applied. This yields at least $(4s' - 5) + (4s'' - 5) = 4s - 10$ crossings, which together with the 5 crossings previously identified, give the $4s - 5$ crossings required to complete the inductive step.

Now suppose that either e_1 or e_2 is incident with a leaf vertex. It is readily checked that then both e_1 and e_2 are incident with leaf vertices, and, moreover, that $p_{j_0}, p_{j_1}, p_{j_2}$ appear consecutively in $\mathcal{P}$ (moreover, $j_2 = 3t - 1$) , so that p_{j_1} is the leaf vertex incident with e_1 and p_{j_2} is the leaf vertex incident with e_2. Hence in this case we might as well assume that (iii) holds. Thus we complete the proof by analyzing the case in which (iii) holds.

Suppose finally that (iii) holds. Remove e_1 and e_2. This removes at least 4 crossings, since e_0 only crosses e_1 and e_2. By contracting e_0 (along with all its incident edges), collapsing p_0 and p_{j_0} and replacing them by a vertex placed in any point of e_0, we obtain a collection of points, of size $3(s-1)$ to which the inductive hypothesis can be applied. This yields at least $4(s-1) - 5 = 4s - 9$ crossings, which together with the 4 crossings previously removed, give the $4s-5$

crossings required to complete the inductive step. This completes the proof of Claim 11. ∎

¿From Claim 11, it is clear that in order to establish the inductive step we need to show that e_0, e_1, e_2 are involved in sufficiently many crossings. Our next statement shows that a large number of crossings is always guaranteed if the end vertices of e_0, e_1, e_2 appear in certain order.

Claim 12 *Suppose that either (a) p_0, p_{i_2}, p_{i_1} appear in $\mathcal{P}$ in the given order; or (b) $p_{j_0}, p_{j_2}, p_{j_1}$ appear in $\mathcal{P}$ in the given order. Then there are at least 3 external crossings.*

Proof. We prove the statement under the assumption that (a) holds. The proof for the case in which (b) holds is totally analogous.

Since p_{i_2} cannot be an immediate successor of p_0, it follows that there is some 1–vertex p_{ℓ_1} such that $0 < \ell_1 < i_2$. Similarly, there is some 0–vertex p_{ℓ_0} such that $i_2 < \ell_0 < i_1$, and there is some 2–vertex p_{ℓ_2} such that $i_1 < \ell_2 < j_0$.

The spanning property of T_1, and the assumption that no 1–edge other than e_1 crosses e_0, imply that there is a T_1–path from p_{ℓ_1} to p_{i_1}. This path must clearly contain an edge (a 1–edge different from e_1) that crosses e_2. This provides an external crossing. Similar arguments show that some 0–edge different from e_0 must cross either e_1 or e_2, (this provides another external crossing), and that some 2–edge different from e_2 must cross e_1 (this provides a third external crossing). This completes the proof of Claim 12. ∎

We are finally ready to establish the inductive step. We analyze separately two cases, depending on whether or not e_1 and e_2 cross each other.

Case 1 *If e_1 and e_2 do not cross each other, then the inductive step follows.*

By (ii) in Claim 11, it suffices to show that there are at least 5 good crossings.

Suppose first that $i_1 < i_2$. It is readily checked that in this case the assumption that e_1 and e_2 do not cross each other implies that $p_{j_0}, p_{j_2}, p_{j_1}$ occur in this order in $\mathcal{P}$. Thus Claim 12 applies, and guarantees the required 5 good crossings (3 external crossings plus 2 internal crossings). Finally, if $i_2 < i_1$, then p_0, p_{i_2}, p_{i_1} occur in this order in $\mathcal{P}$, and again an application of Claim 12 gives the required 5 good crossings. This completes the analysis for Case 1.

Case 2 *If e_1 and e_2 cross each other, then the inductive step follows.*

We claim that, in this case, it suffices to prove the following statements:

(1) There is at least 1 external crossing.

(2) If e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$, then there are at least 2 external crossings.

Indeed, suppose that (1) and (2) hold. Since e_1 and e_2 cross each other, then there are 3 internal crossings. Thus, by (1), there are at least 4 good crossings. If e_0, e_1, e_2 are incident with leaf vertices that appear consecutively in $\mathcal{P}$, this fact together with (i) in Claim 11 imply that the inductive step follows. On the other hand, if e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$, then by (2) there are at least 5 good crossings, and so by (ii) in Claim 11 the inductive step follows.

Thus we finish the analysis of Case 2 (and thus the whole proof) by proving (1) and (2).

Before proving these statements, we make a general observation. Since e_1 and e_2 cross, then the end vertices of e_0, e_1, e_2 appear in $\mathcal{P}$ either in the order $p_0, p_{i_1}, p_{i_2}, p_{j_0}, p_{j_1}, p_{j_2}$ or in the order $p_0, p_{i_2}, p_{i_1}, p_{j_0}, p_{j_2}, p_{j_1}$. In the latter case, Claim 12 applies, in which case both (1) and (2) follow.

Therefore for proving (1) and (2) we may assume that $p_0, p_{i_1}, p_{i_2}, p_{j_0}, p_{j_1}, p_{j_2}$ appear in $\mathcal{P}$ in the given order.

One word on terminology. If p_r, p_t are vertices in $\mathcal{P}$ such that $0 \leq r < t \leq 3t-1$, then the *segment* $[p_r, p_t]$ is the (possibly empty) set $\{p_{r+1}, p_{r+2}, \ldots, p_{t-1}\}$.

Proof of (1)

Note that, since $j_0 \geq 6$, it follows that at least one of the segments $[p_0, p_{i_1}]$, $[p_{i_1}, p_{i_2}]$, $[p_{i_2}, p_{j_0}]$ is nonempty. Note that any such nonempty segment contains at least one 0–vertex, one 1–vertex, and one 2–vertex. Suppose for instance that $[p_{i_2}, p_{j_0}]$ is nonempty. Thus there is a 1–vertex p_{ℓ_1} such that $i_2 < \ell_1 < j_0$. The path in T_1 that joins p_{ℓ_1} and p_{j_1} must clearly cross e_2. Thus, some 1–edge other than e_1 crosses e_2. This provides an external crossing. A similar argument shows that if $[p_0, p_{i_1}]$ is nonempty, then some 2–edge other than e_2 crosses e_1. Yet another application of the same argument shows that if $[p_{i_1}, p_{i_2}]$ is nonempty, then some 0–edge other than e_0 crosses either e_1 or e_2. Therefore, in either case we obtain an external crossing, as required.

Proof of (2)

First we claim that if $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ do not appear consecutively in $\mathcal{P}$, then there are at least 2 external crossings, in which case (2) immediately follows.

Suppose that $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ do not appear consecutively in $\mathcal{P}$, that is, one of the segments $[p_{j_0}, p_{j_1}]$, $[p_{j_1}, p_{j_2}]$, $[p_{j_2}, p_0]$ is nonempty. An argument totally analogous to the one used in the proof of (1) shows that the nonemptiness of any such segments guarantees the existence of an external crossing.

Thus, if $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ do not appear consecutively in $\mathcal{P}$, then there are at least two external crossings. Indeed, the crossing identified in the previous paragraph, plus the crossing obtained in the proof of (1), are clearly distinct. Thus in this case we have the required 2 external crossings.

In view of (1) and this discussion, in order to complete the proof of (2) it suffices to show the following: if $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ appear consecutively in $\mathcal{P}$, and e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$, then

then there are at least 2 external crossings. The rest of the proof is devoted to show this statement.

First we observe that since $p_{j_0}, p_{j_1}, p_{j_2}, p_0$ appear consecutively in $\mathcal{P}$, it follows that p_{j_1} and p_{j_2} are leaf vertices of e_1 and e_2, respectively. Thus the assumption that e_0, e_1, e_2 are not incident with leaf vertices that appear consecutively in $\mathcal{P}$ implies that e_0 is not incident with leaf vertices. That is, neither p_0 nor p_{j_0} is a leaf vertex.

Suppose that the segment $[p_0, p_{i_1}]$ is nonempty. Then it contains a 2–vertex, and an argument analogous to the one used in the proof of (1) shows that this implies that there is an external crossing of a 2–edge other than e_2 with e_1.

Similarly, if $[p_{i_2}, p_{j_0}]$ is nonempty, then it contains a 1–vertex, and so there is an external crossing of a 1–edge other than e_1 with e_2. By a similar token, if $[p_{i_1}, p_{i_2}]$ is nonempty, then it contains a 0–vertex, and so there is an external crossing of a 0–edge other than e_0 with either e_1 or e_2.

These arguments show that if at least two of the segments $[p_0, p_{i_1}]$, $[p_{i_1}, p_{i_2}]$, $[p_{i_2}, p_{j_0}]$ are nonempty, then there are at least 2 external crossings, as required.

Thus for the rest of the proof we assume that exactly one of the segments $[p_0, p_{i_1}]$, $[p_{i_2}, p_{j_0}]$, $[p_{i_1}, p_{i_2}]$ is nonempty.

Suppose that $[p_0, p_{i_1}]$ is nonempty and both $[p_{i_1}, p_{i_2}]$ and $[p_{i_2}, p_{j_0}]$ are empty. Then $[p_0, p_{i_1}]$ must contain some 0–vertex. Moreover, $[p_0, p_{i_1}]$ must contain some 0–vertex that is connected to p_{j_0} via a T_0–path that does not contain p_0, as otherwise p_{j_0} would be a leaf. This implies that some 0–edge other than e_0 crosses *both* e_1 and e_2. This provides the two required external crossings.

An analogous argument takes care of the case in which $[p_{i_2}, p_{j_0}]$ is nonempty and both $[p_0, p_{i_1}]$ and $[p_{i_1}, p_{i_2}]$ are empty.

Thus we finish the proof by dealing with the case in which $[p_{i_1}, p_{i_2}]$ is nonempty and both $[p_0, p_{i_1}]$ and $[p_{i_2}, p_{j_0}]$ are empty. In this case, $[p_{i_1}, p_{i_2}]$ must contain a 0–vertex connected to p_0 via a T_0–path that does not contain p_{j_0}, as otherwise p_0 would be a leaf. For a similar reason, $[p_{i_1}, p_{i_2}]$ must contain a 0–vertex connected to p_{j_0} via a T_0–path that does not contain p_0. One of these paths must cross e_1, and the other one must cross e_2. This gives the two required external crossings. ∎

3 Non alternating three–coloured collections: proof of Theorem 7

The heart of the proof of Theorem 7 is the following statement.

Proposition 13 *Let $\mathcal{P} = \{P_0, P_1, P_2\}$ be a good collection of point sets such that $P_0 \cup P_1 \cup P_2$ is in convex position, $|P_0| \le |P_1| \le |P_2|$, and $n := |P_0 \cup P_1 \cup P_2| \ge 6$. Suppose further that $\mathcal{P}$ is not alternating. Then there is a corresponding set of trees $\mathcal{T}$ for $\mathcal{P}$ such that* $\mathrm{int}(\mathcal{T}) < (4/3)n - 5$.

Note that Theorem 7, whose proof is the goal of this section, is an immediate consequence of Proposition 13.

Proof of Theorem 7. It follows from Proposition 13: by relabelling P_0, P_1, P_2, if necessary, it can be assumed without loss of generality that $|P_0| \leq |P_1| \leq |P_2|$, so that Proposition 13 applies. ∎

The rest of this section is thus devoted to the proof of Proposition 13.

Proof of Proposition 13. For the sake of clarity, we break the proof into four steps.

STEP 1 *Construction of a corresponding set of trees $\mathcal{T} = \{T_0, T_1, T_2\}$ for $\mathcal{P} = \{P_0, P_1, P_2\}$, given* a starting point $p_0 \in P_0$.

Let p_0 be any point in P_0 (we call p_0 the *starting point* of the construction). Label the points of $P := P_0 \cup P_1 \cup P_2$ as $p_0, p_1, \ldots, p_{n-1}$, so that the p_i's occur in the given clockwise cyclic order in the convex hull of P. For $r = 1, 2$, let i_r be the least integer such that p_{i_r} is in P_r, and let j_r be the largest integer such that p_{j_r} is in P_r.

For $r = 1, 2$, let C_r denote the convex polygon with vertex set P_r. The convexity of C_1 and C_2 implies that each edge of C_1 intersects at most twice (the edges in) C_2.

Let T_1 be the tree (moreover, path) that results from removing from C_1 the edge $p_{i_1}p_{j_1}$. Similarly, let T_2 be the tree (path) that results from removing from C_2 the edge $p_{i_2}p_{j_2}$. Finally, let T_0 be the star whose vertex set is P_0, and whose vertex of degree $|P_0| - 1$ is p_0.

Clearly, $\mathcal{T} := \{T_0, T_1, T_2\}$ is a corresponding set of trees for $\mathcal{P}$.

STEP 2 $\mathrm{int}(\mathcal{T}) \leq (4/3)n - 5$, *independently of the choice of the point p_0.*

Let α denote the number of edges of T_1 that intersect at most one edge of T_2. Thus, the number of intersections among edges in $T_1 \cup T_2$ is at most $2(|P_1| - 1) - \alpha$. If $\alpha = 0$, then each edge of T_1 intersects exactly two edges in T_2, and so in this case $|P_2| \geq |P_1| + 1$.

From the construction of T_0, T_1, and T_2 it follows that each edge of T_0 crosses at most one edge of T_1 and at most one edge of T_2. Therefore the number of intersections of edges that involve an edge in T_0 is at most $2(|P_0| - 1)$.

Hence, $\mathrm{int}(\mathcal{T}) \leq 2(|P_1| - 1) - \alpha + 2(|P_0| - 1) = 2(|P_0| + |P_1|) - (\alpha + 4)$.

Thus, to complete Step 1 it suffices to show that $2(|P_0| + |P_1|) - (\alpha + 4) \leq (4/3)n - 5$.

We remark that $n = |P_0| + |P_1| + |P_2|$.

We claim that strict inequality holds if either (i) $|P_0| \leq |P_2| - 2$ or (ii) $\alpha = 0$.

For suppose that $|P_0| \leq |P_2| - 2$. Then, s ince $|P_1| \leq |P_2|$, it follows that $|P_0| + |P_1| \leq 2|P_2| - 2$. A straightforward manipulation shows that then $2(|P_0| + |P_1|) - (\alpha + 4) < (4/3)(|P_0| + |P_1| + |P_2|) - 5$.

Now suppose that $\alpha = 0$. Then $|P_2| \geq |P_1| + 1$. On the other hand, since $|P_0| \leq |P_1|$, and $|P_2| - |P_0| \leq 1$, it follows that $|P_2| - 1 = |P_0| = |P_1|$. A straightforward manipulation then yields the strict inequality $2(|P_0|+|P_1|) - (\alpha+4) < (4/3)(|P_0|+|P_1|+|P_2|) - 5$.

Thus either we have strict inequality or $|P_2| - |P_0| \leq 1$ and $\alpha = 0$.

It remains to check the case $\alpha \geq 1$, that is, $-(\alpha+4) \leq -5$. It is an easy observation that $2(|P_0|+|P_1|) \leq (4/3)(|P_0|+|P_1|+|P_2|)$, and so a trivial manipulation shows that $2(|P_0|+|P_1|) - (\alpha+4) \leq (4/3)(|P_0|+|P_1|+|P_2|) - 5$.

Step 3 *If $|P_0| < |P_2|$, then $\mathrm{int}(\mathcal{T}) < (4/3)n - 5$, independently of the choice of the point p_0.*

Suppose that $|P_0| < |P_2|$. From the case analysis above, it follows that $\mathrm{int}(\mathcal{T}) \leq (4/3)n - 5$, and equality can hold only if $|P_2| - |P_0| \leq 1$ and $\alpha \geq 1$. Thus either strict inequality holds (in which case Step 3 is done) or we can assume $|P_0| < |P_2|$, $|P_2| - |P_0| \leq 1$, and $\alpha \geq 1$. But then $2(|P_0|+|P_1|) < 4|P_2|$, and $-(\alpha+4) \leq -5$. These readily imply that $2(|P_0|+|P_1|) - (\alpha+4) < (4/3)n - 5$.

Step 4 *After relabelling P_0, P_1, P_2, if necessary, there is a choice of the starting point p_0 such that the constructed $\mathcal{T} = \{T_0, T_1, T_2\}$ satisfies $\mathrm{int}(\mathcal{T}) < (4/3)n - 5$.*

First we note that, by Step 3, we may assume that $|P_0| = |P_2|$. Since $|P_0| \leq |P_1| \leq |P_2|$ by assumption, it follows that $|P_0| = |P_1| = |P_2|$.

It is easy to check that, since $\{P_0, P_1, P_2\}$ is not alternating and $|P_0| = |P_1| = |P_2|$, then either (i) there are two points p, q in the same P_i that appear consecutively in the convex hull of P; or (ii) there are points p, q, r that appear consecutively in the convex hull of P (in this clockwise order), such that p and r belong to the same P_i and q belongs to a $P_j \neq P_i$. We examine these cases separately.

Suppose that (i) holds. By relabelling P_0, P_1, P_2 if necessary, we may assume that $p, q \in P_0$. Thus we set $p_0 = p$ (this implies $p_1 = q$), and obtain a corresponding set of trees $\mathcal{T} = \{T_0, T_1, T_2\}$. Recall that the number of intersections of edges of T_1 with edges of T_2 is at most $2(|P_1| - 1)$, and that each edge of T_0 is crossed at most twice. Since the edge p_0p_1 intersects no edge, it follows that T_0 has at most $2(|P_0| - 2)$ intersections with edges in $T_1 \cup T_2$, and so $\mathrm{int}(\mathcal{T}) \leq 2(|P_0|+|P_1|) - 6 = 4|P_0| - 6$. Since $|P_0| = n/3$, we obtain $\mathrm{int}(\mathcal{T}) \leq (4/3)n - 6 < (4/3)n - 5$, as required.

Suppose finally that (ii) holds. By relabelling P_0, P_1, P_2 if necessary, we may assume that $q \in P_0$ and $p, r \in P_1$. We thus proceed to construct $\mathcal{T} = \{T_0, T_1, T_2\}$ with the starting point $p_0 = q$.

An easy counting argument shows that there is an edge of T_1 that intersects no edge of T_2, so that there are at most $2(|P_1| - 2)$ intersections between T_1 and T_2. On the other hand, the edges of T_0 have at most $2(|P_0| - 1)$ intersections with the edges of $T_1 \cup T_2$, and so $\mathrm{int}(\mathcal{T}) \leq 2(|P_0|+|P_1|) - 6 = 4|P_0| - 6$. Since $|P_0| = n/3$, we obtain $\mathrm{int}(\mathcal{T}) \leq (4/3)n - 6 < (4/3)n - 5$, as required. ∎

4 Parity issues of Conjecture 5: proof of Theorem 8

Let $k \geq 3$ be an odd integer. We assume that Conjecture 5 holds for k, and will show that it follows that it also holds for $k+1$.

Let $\mathcal{P} = \{P_0, P_1, \ldots, P_k\}$ be an alternating collection, where $k+1$ is even (note that $\mathcal{P}$ has $k+1$ point sets P_i), and let $\mathcal{T} = \{T_0, T_1, \ldots, T_k\}$ be a corresponding set of trees.

We need to show that the edges in $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$ intersect in at least $(3k^2/4 - k)((n/k) - 1) - k(k-2)/4$ points.

Since by assumption Conjecture 5 holds for k, it follows that, for each $i \in \{0, \ldots, k\}$, the edges in $(T_0 \cup \cdots \cup T_k) \setminus \{T_i\}$ intersect at least $(3(k-1)^2/4 + (k-1)/2)(((n/k) - 1) - (k-1)^2/4$ times.

An elementary counting argument then shows that the edges in $T_0 \cup \cdots \cup T_k$ intersect at least $\binom{k+1}{k}\big[((3(k-1)^2/4 + (k-1)/2)((n/k) - 1) - (k-1)^2/4)\big]/(k-2)$ times. A straightforward manipulation shows that this number is *exactly* $(3(k+1)^2/4 - (k+1))((n/k) - 1) - (k+1)((k+1) - 2)/4$, as required. ∎

5 Multicoloured alternating collections: proof of Theorem 10

Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$ be alternating, where $k > 3$. Let $P_{r_1}, P_{r_2}, P_{r_3}$ be any three distinct collections of $\mathcal{P}$. By Theorem 6, there are at least $(4/3)(3n/k)$ -5 intersections that involve only edges in $T_{r_1} \cup T_{r_2} \cup T_{r_3}$.

Since there are $\binom{k}{3}$ ways to choose such $P_{r_1}, P_{r_2}, P_{r_3}$, an elementary counting argument shows that the total number of intersections of edges in $T_0 \cup T_1 \cup \cdots \cup T_{k-1}$ is at least

$$\binom{k}{3}\frac{(4/3)(3n/k) - 5}{k-2}$$

(here we divide by $k-2$ since each P_r is in $k-2$ different 3–collections $\{P_{r_1}, P_{r_2}, P_{r_3}\}$, so that each intersection gets overcounted $k-2$ times).

A trivial manipulation shows that this expression equals $(2/3)(k-1)n - 5k(k-1)/6$, as claimed. ∎

6 Concluding Remarks

As we mentioned in Section 1, the analysis of good collections whose union is in general position is motivated by the drive to test the tightness of Theorem 1. Our Theorem 7 then shows that our best bet is to focus on alternating collections.

For alternating collections, Kano et al. put forward a general conjecture, namely Conjecture 5. In this paper we have settled this conjecture for $k = 3$ and 4. Naturally, the next step would be to try to settle the conjecture for larger values of k, aiming in the process to gain some insight into the general problem.

Proving Conjecture 5 true for larger values of k would automatically imply a better tightness estimate for Theorem 1. However, one must keep in mind that even settling Conjecture 5 for every k would not imply that Theorem 1 is (asymptotically) tight for each k. This approach to the problem of testing the tightness of Theorem 1 has a natural limit (namely a factor of 4/3), as the next result shows.

Theorem 14 *Suppose that Conjecture 5 is true for some odd integer* $k_0 \geq 3$. *Then, for every fixed* $k \geq k_0$, *the bound in Theorem 1 is asymptotically within a factor of* $4k_0/(3k_0 - 1)$ *from the optimal solution.* ∎

The proof of this statement is a straightforward counting argument.

Theorem 14 suggests that as a next step it makes sense to combine an effort to prove Conjecture 5 for $k \geq 5$ with an attempt to improve on Theorem 1. This last direction would very likely include a further exploration on the case in which the set $P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ is not necessarily in convex position.

References

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