# Order types and cross-sections of line arrangements in $\mathbb{R}^3$

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## Abstract

We consider sets $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ of $n$ labeled lines in general position in $\mathbb{R}^3$, and study the order types of point sets $\{p_1, \ldots, p_n\}$ that stem from the intersections of the lines in $\mathcal{L}$ with (directed) planes $\Pi$, not parallel to any line of $\mathcal{L}$, i.e., the proper cross-sections of $\mathcal{L}$. As a main result we show that the number of different order types that can be obtained as cross-sections of $\mathcal{L}$ is $\mathcal{O}(n^9)$, and that this bound is tight.

## 1 Introduction

Let $\mathcal{L}$ be an arrangement of $n$ lines in $\mathbb{R}^3$, labeled $\ell_1, \ldots, \ell_n$. We say that $\mathcal{L}$ is in *general position* if no two lines of $\mathcal{L}$ are co-planar. Observe that if $\mathcal{L}$ is in general position then its elements are pairwise disjoint. In the following, all considered sets of lines $\mathcal{L}$ are assumed to be in general position. For a directed plane $\Pi$ (not parallel to any of the lines in $\mathcal{L}$), consider the intersections of the lines of $\mathcal{L}$ with $\Pi$, denoted $\mathcal{L}_\Pi = \{p_1, \ldots, p_n\}$, where $p_i = \ell_i \cap \Pi$ for every $i$. We call $\mathcal{L}_\Pi$ a *cross-section (of $\mathcal{L}$ induced by $\Pi$)*. A *general* cross-section is a cross-section in which the point set $\mathcal{L}_\Pi$ is in general position (i.e. no three points are on a common line). Note that the statement "the order type of a cross-section" is not properly defined for non-general cross-sections. Thus, for the questions considered in this paper, we will mostly refer to general cross-sections. However, we will also encounter non-general cross-sections.

Arrangements of lines in $\mathbb{R}^3$ have been considered in the past in the computational geometry community, mostly trying to describe the " combinatorial position" of the arrangement. For example, Chazelle et al. [4] provided, among other results, a tight $\Theta(n^2)$ bound on the complexity of the space of all lines passing above all the $n$ given lines and satisfying a certain orientation consistency constraint, and McKenna and O'Rourke [10] studied equivalence classes of sets of lines that are "entangled" in $n$ lines.

A natural question in our scenario is to ask how many combinatorially different sets can be obtained as cross-sections of $\mathcal{L}$ or, in other words, how many different order types can be obtained from the cross-sections of $\mathcal{L}$. We prove the answer to this problem to be $O(n^9)$, and that this bound is achievable.

On the other hand, problems related to points moving in the plane, each one along a line, with its own constant speed, have been extensively studied [1, 2, 3, 6, 7, 8, 9]. Now, if $P$ is the set of points, their movement can be modeled using a set of lines $\mathcal{L}$ in $\mathbb{R}^3$ and a horizontal plane that sweeps the space, moving vertically at constant speed. Each moving point $p$ is mapped to the line $\ell_p$ where $(x, y, z)$ is a point of $\ell_p$ if and only if the point $p$ has coordinates $(x, y)$ at the instant of time $t = z$. In other words, the position of all elements of $P$ at time $t$ is precisely the cross-section $\mathcal{L}_{\{z=t\}}$.

In our study of cross-sections we consider this particular situation, and our results translate into that the number of combinatorially different points sets (this is, different order types) obtained from $n$ points $p_1, \ldots, p_n$ moving in the plane along lines, at constant speed, is $O(n^3)$, which is tight.

## 2 Lines intersecting three lines of $\mathcal{L}$

Any three lines $\ell_i, \ell_j, \ell_k$ in $\mathbb{R}^3$ are contained in some quadric. If the lines are in general position, the quadric is unique, and it is either a *hyperbolic paraboloid* $\mathcal{P}$ or a *one-sheet hyperboloid* $\mathcal{H}$; see Figure 1. In both cases, the three lines form a subset of one of the two families of generatrices of the corresponding quadric. As all hyperboloids we consider are one-sheet hyperboloids, in the following we omit the specification "one sheet".

**Observation 1** *For any three lines $\ell_i, \ell_j, \ell_k$ in general position, there is an infinite set of lines intersecting $\ell_i$, $\ell_j$, and $\ell_k$, namely, the family of generatrices of the quadric defined by $\{\ell_i, \ell_j, \ell_k\}$ that does not contain $\{\ell_i, \ell_j, \ell_k\}$.*

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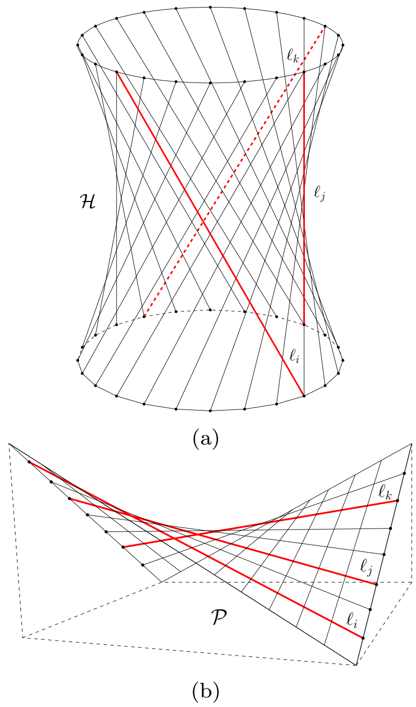


Figure 1: Three lines $\ell_i$, $\ell_j$, and $\ell_k$ in general position define either (a) a one-sheet hyperboloid $\mathcal{H}$, or (b) a hyperbolic paraboloid $\mathcal{P}$.

**Observation 2** *For any four lines $\ell_i$, $\ell_j$, $\ell_k$, $\ell_m$ in general position, either at most two lines intersect all of $\{\ell_i, \ell_j, \ell_k, \ell_m\}$, or every line intersecting three lines of $\{\ell_i, \ell_j, \ell_k, \ell_m\}$ also intersects the fourth (and $\ell_i$, $\ell_j$, $\ell_k$, and $\ell_m$ are generatrices of the same quadric).*

Note that a set $\mathcal{L}$ of $n$ lines from one family of generatrices of a hyperboloid (or a hyperbolic paraboloid) is in general position, as no two lines of $\mathcal{L}$ are co-planar. Still, $\mathcal{L}$ is in some sense very degenerate: any line $\ell$ intersecting three lines of $\mathcal{L}$ intersects all lines of $\mathcal{L}$. Hence, we say that an arrangement $\mathcal{L}$ of $n$ lines in $\mathbb{R}^3$ is in *strong general position* if no two lines of $\mathcal{L}$ are co-planar, no three lines of $\mathcal{L}$ lie in pairwise parallel planes, and no four lines of $\mathcal{L}$ lie on the same quadric. For a set $\mathcal{L}$ of lines in strong general position, every subset of size three defines a different hyperboloid.

## 3 Cross-sections and order types

Given a (labeled) point set $P$ in the plane in general position, every ordered triple of points of $P$ has an orientation (counter-clockwise or clockwise). All triple orientations together form the *(labeled) order type* of $P$. Two point sets have the same order type if all triple orientations coincide. Order types are means for grouping the infinite set of different $n$-point sets in the plane into a finite set of equivalence-classes, and most combinatorial properties of point sets only depend on their order types. In a common variation the point sets are not labeled. Then two point sets have the same order type if there exists a labeling of the two sets such that all triple orientations coincide. Moreover, often also reflection of one point set is allowed, that is, all triple orientations are inverted. See the survey [5] for more details, other variants, and the history of order types. In this paper we only use labeled order types and distinguish between a point set and its reflected copy. For the sake of brevity we will simply use the term order type throughout.

### 3.1 Sectioning $\mathcal{L}$ with parallel planes

For an arrangement $\mathcal{L}$ of $n$ lines in $\mathbb{R}^3$, consider a direction $\vec{\ell}$ not orthogonal to any line of $\mathcal{L}$ and the sequence of order types that is obtained when *sweeping $\mathcal{L}$ in the direction of $\vec{\ell}$*, that is, the sequence of order types obtained from the cross-sections $\mathcal{L}_\Pi$, where the planes $\Pi$ are orthogonal to and directed according to $\vec{\ell}$ and ordered with respect to their intersection points with $\vec{\ell}$. The following theorem gives an upper bound on the length of this sequence. Its proof, based in the fact that every triple of points changes its orientation at most twice, as well as the proof of Proposition 2, can be found in the full version of this work.

**Theorem 1** *For any arrangement $\mathcal{L}$ of $n$ lines in general position and any directed line $\vec{\ell}$ that is not orthogonal to any line of $\mathcal{L}$, the number of different order types obtained from cross-sections $\mathcal{L}_\Pi$ induced by planes $\Pi$ orthogonal to $\vec{\ell}$ is at most $2\binom{n}{3}$.*

**Proposition 2** *Consider an arrangement $\mathcal{L}$ of $n$ lines in general position where no three lines of $\mathcal{L}$ lie in pairwise parallel planes, a directed line $\vec{\ell}$ that is not orthogonal to any line of $\mathcal{L}$, and cross-sections of $\mathcal{L}$ induced by planes $\Pi$ orthogonal to $\vec{\ell}$. Then the order type of $\mathcal{L}_\Pi = \{p_1, \ldots, p_n\}$ for $\Pi \to -\infty$ (with respect to its intersection with $\vec{\ell}$) is the same as the one for $\Pi \to +\infty$.*

Whenever a triple changes its orientation during a sweep, there is a plane $\Pi$ in which the triple shows up as collinear points, implying that $\Pi$ contains a generatrix of the quadric defined by the line triple. If this quadric is a hyperboloid, then $\Pi$ also contains a point of the waist of the hyperboloid. Thus, for a given directed line $\vec{\ell}$ and an arrangement $\mathcal{L}$ with no three lines in parallel planes, there exist planes $\Pi^+$ and $\Pi^-$ orthogonal to $\vec{\ell}$ such that (1) the order type $\vec{\mathbf{o}}$ of the cross-section of $\mathcal{L}$ induced by $\Pi^+$ is identical to the order type induced by $\Pi^-$ and (2) the order type of the cross-section of $\mathcal{L}$ induced by any plane $\Pi$ that is orthogonal to $\vec{\ell}$ and does not lie between $\Pi^+$ and $\Pi^-$ is $\vec{\mathbf{o}}$ as well. In other words, when sweeping $\mathcal{L}$ in the direction of $\vec{\ell}$, all order type changes occur between $\Pi^-$ and $\Pi^+$. We denote $\vec{\mathbf{o}}$ as *stable order type (of $\mathcal{L}$ with respect to $\vec{\ell}$).*

Note that there exist sets $\mathcal{L}$ and directions $\vec{\ell}$ such that sweeping $\mathcal{L}$ in the direction of $\vec{\ell}$ gives only one single order type. Also, there are sets $\mathcal{L}$ such that for any direction $\vec{\ell}$ there exists a triple $\ell_i, \ell_j, \ell_k \subset \mathcal{L}$ such that the orientation of $\ell_i, \ell_j, \ell_k$ is constant with respect to $\vec{\ell}$. Thus there is obviously no guaranteed lower bound on the number of different order types obtained from sweeping a line arrangement in some fixed direction. On the other hand, the following theorem states that the upper bound from Theorem 1 is asymptotically tight.

**Theorem 3** *There exist sets $\mathcal{L}$ and directions $\vec{\ell}$ such that the number of different order types obtained from cross-sections of $\mathcal{L}$ induced by planes orthogonal to $\vec{\ell}$ is at least $\binom{n}{3}+1$.*

**Proof.** The proof is by construction of a set $\mathcal{L}$ and a direction $\vec{\ell}$ such that when sweeping $\mathcal{L}$ in the direction of $\vec{\ell}$, all triples $\ell_i, \ell_j, \ell_k \subseteq \mathcal{L}$ change their orientation once before any of them changes its orientation a second time. Consider two parallel planes $\Pi_1$ and $\Pi_2$ with circles $C_1 \in \Pi_1$ and $C_2 \in \Pi_2$, with different radii, and with centers $m_1$ and $m_2$, respectively. In $\Pi_1$, arrange points with labels $1, \ldots, n$ in clockwise order on one half of $C_1$. In $\Pi_2$, arrange points with the same labels $1, \ldots, n$ in counterclockwise order on one half of $C_2$. Let $\mathcal{L}$ be the set of lines connecting $p_i$ from $\Pi_1$ with $p_i$ from $\Pi_2$, for $1 \leq i \leq n$, and let $\vec{\ell}$ be the line connecting $m_1$ and $m_2$; see Figure 2.

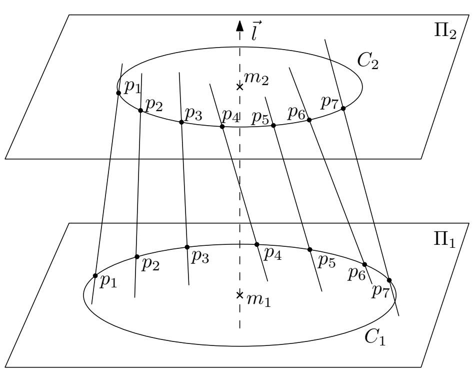


Figure 2: A set of lines and a direction where $\binom{n}{3}+1$ different order types are encountered with planes orthogonal to $\vec{\ell}$.

As every point triple $p_ip_jp_k$ has inverse orientations in $\Pi_1$ and $\Pi_2$, respectively, and as a triple changes its orientation at most twice during a sweep, exactly one such change occurs between $\Pi_1$ and $\Pi_2$. Arranging the points on $C_1$ and $C_2$ such that no two triples change their orientation simultaneously during the sweep gives $\binom{n}{3}+1$ order types when sweeping from $\Pi_1$ to $\Pi_2$. $\square$

**Corollary 4** *The maximum number of different order types obtained from cross-sections of a set $\mathcal{L}$ of $n$ lines in general position by sweeping $\mathcal{L}$ in an arbitrary but fixed direction $\vec{\ell}$ not orthogonal to any line of $\mathcal{L}$ is $\Theta(n^3)$.*

**Corollary 5** *Let $S$ be a set of $n$ points $\{p_1, \ldots, p_n\}$ in the plane, such that every $p_i$ moves along a line $r_i$ with constant speed $\vec{v}_i$, for $i = 1, \ldots, n$. The maximum number of different order types obtained from these points is $\Theta(n^3)$, which is tight.*

### 3.2 Sectioning $\mathcal{L}$ with general planes

Knowing that the number of order types that can be obtained from sweeping an arrangement of lines in some fixed direction is at leas 1 and at most $\Theta(n^3)$, we now consider the number of stable order types.

**Theorem 6** *For any set $\mathcal{L}$ with $n \geq 4$ lines with no three lines of $\mathcal{L}$ being in pairwise parallel planes, the number of different stable order types of $\mathcal{L}$ is exactly the number of cells in an arrangement of $n$ great circles on the sphere and thus $\Theta(n^2)$.*

**Proof.** For any $1 \leq i < j < k \leq n$, let $\mathcal{H}_{ijk}$ be the hyperboloid defined by the lines $\ell_i, \ell_j, \ell_k \subset \mathcal{L}$. Consider a ball containing the waists of all these hyperboloids in its interior, and let $S$ be the sphere which bounds this ball. Then any plane which does not intersect $S$ and is not parallel to any of the lines in $\mathcal{L}$ induces a cross-section with a stable order type. Further, as translating a plane without intersecting $S$ does not induce any order type changes, all these stable order types are also encountered by cross-sections induced by planes tangent to $S$. On $S$, consider for every line $\ell_i \in \mathcal{L}$ the great circle $C_i$ which is in the plane orthogonal to $\ell_i$ that contains the center of $S$. Further, consider a plane $\Pi$ that is tangent to $S$ and move in some way that it stays tangent to $S$.

In general, when moving a generic plane, an orientation change of a triple in the cross-section can either come from the three points becoming collinear, or from the plane traversing a position parallel to a line $\ell_i$ of $\mathcal{L}$ which induces a $\pm\infty$ change of the according point $p_i$ (and thus an orientation change of all triples which involve $p_i$). In the situation of $\Pi$ staying tangent to (or outside of) $S$, the first cannot happen.

The second situation happens exactly when $\Pi$ traverses a great circle $C_i$. Thus, within one cell of the great circle arrangement $\mathcal{C} = \{C_i\}$ on $S$, all tangent planes induce the same order type, implying that the number of different order types induced by planes not intersecting $S$ is at most the number of cells in the great circle arrangement $\mathcal{C}$.

What remains to show is that no two different cells of $\mathcal{C}$ represent the same stable order type. To this end, consider two arbitrary cells $f_1$ and $f_2$ of $\mathcal{C}$. The shortest path on $S$ between $f_1$ and $f_2$ crosses every circle $C_i$ at most once. Let $1 \leq k \leq n$ be the number of circles that are crossed by this path $P$ and $n-k$ be the number of non-crossed circles . We distinguish four cases for $k$.

*Case* $1 \leq k \leq n-3$ Consider a circle $C_i$ that is crossed by $P$ and three circles $C_j$, $C_k$, and $C_l$ that are not crossed by $P$. Then the triple $p_ip_jp_k$ changes its orientation exactly once (when traversing $C_i$) while the triple $p_jp_kp_l$ does not ever change its orientation. Hence, the order type induced by planes tangent to $f_1$ cannot be identical to the order type induced by planes tangent to $f_2$.

*Case* $2 \leq k \leq n-2$ Consider two circles $C_i$ and $C_j$ that are crossed by $P$ and two circles $C_k$ and $C_l$ that are not crossed by $P$. The triple $p_ip_jp_k$ changes its orientation exactly twice, while the triple $p_jp_kp_l$ changes its orientation only once. Hence $p_jp_kp_l$ has different orientations in the order types represented by $f_1$ and $f_2$, respectively, while $p_ip_jp_k$ has the same orientation.

*Case* $3 \leq k \leq n-1$ This case is reverse to the first one. Let $C_i$, $C_j$, and $C_k$ be three circles crossed by $P$ and $C_l$ be a circle not crossed by $P$. Then $p_ip_jp_k$ changes has different orientations in the order types corresponding to $f_1$ and $f_2$, respectively, while $p_ip_jp_l$ has the same orientation in both.

*Case* $k = n$ If all great circles are crossed by $P$ then all triples change their orientation exactly three times along $P$ and thus all orientations are reversed between the order type corresponding to $f_1$ and the one corresponding to $f_2$. □

**Proposition 7** *There exist sets $\mathcal{L}$ of $n$ lines in general position with no three of them being parallel to the same plane for which the total number of order types of general cross-sections is only $\Theta(n^2)$.*

One example obtaining the $\Theta(n^2)$ bound is a subset $\mathcal{L}$ of one family of generatrices of a hyperboloid $\mathcal{H}$. Due to space constraints, the proof is deferred to the full version.

**Corollary 8** *The minimum number of different order types obtainable from a set of lines in general position with no three of them being in parallel planes is $\Theta(n^2)$.*

Note that the upper bound from Proposition 7 heavily uses that the considered lines all lie on the same hyperboloid. So next we consider the contrary, i.e., line arrangements in strong general position.

**Lemma 9** *Let $\mathcal{L}$ be a set of at least three lines in strong general position. For any subset $\{\ell_i, \ell_j, \ell_k\} \subseteq \mathcal{L}$ there exists a plane $\Pi$, such that $\Pi \cap \{\ell_i, \ell_j, \ell_k\}$ is collinear and all other triples of $\Pi \cap \mathcal{L}$ are in general position.*

**Proof.** Let $\mathcal{H}$ be the hyperboloid generated by $\{\ell_i, \ell_j, \ell_k\}$. Since $\mathcal{L}$ is in strong general position every line of $\mathcal{L}$ intersects $\mathcal{H}$ in at most two points, we call all such points of $\mathcal{H}$ *bad* points. As the the number of bad points of $\mathcal{H}$ is finite, we can choose a line $\ell$ intersecting $\ell_i, \ell_j$ and $\ell_k$ and not containing any bad points. Let $\Pi$ be a plane containing $\ell$. If another triple of points of $\mathcal{L} \cap \Pi$ is collinear, by choice of $\ell$, its points cannot all lie on $\ell$. Therefore, if necessary, we can destroy any other collinearities by rotating $\Pi$ slightly around $\ell$. □

**Theorem 10** *Let $\mathcal{L}$ be a set of $n$ lines in strong general position. The minimum number of different order types obtainable from $\mathcal{L}$ by general cross-sections is $\Omega(n^3)$.*

**Proof.** Let $\mathcal{T}$ be the set of all triples of lines of $\mathcal{L}$. Choose an arbitrary but fixed ordering $\langle \mathcal{T} \rangle$ of $\mathcal{T}$. To every general cross-section $\mathcal{L}_\Pi$ assign a vector $v_\Pi \in \{+1, -1\}^{\binom{n}{3}}$. The $i$-th entry of $v_\Pi$ corresponds to the orientation of the intersection of the $i$-th triple of $\langle \mathcal{T} \rangle$ with $\Pi$ (where $+1$ means counter-clockwise orientation and $-1$ clockwise orientation). Let $V$ be the set of all such vectors. Note that the number of different order types obtainable from $\mathcal{L}$ by general cross-sections is equal to $|V|$.

Let $v_\Pi[i]$ be the vector consisting of the first $i$ coordinates of $v_\Pi$, and let $V_i$ be the set of all such vectors of length $i$. Clearly $|V_1| \geq 1$. We claim that for $i > 1$, $|V_i| \geq |V_{i-1}| + 1$. Note that for each vector of $V_{i-1}$, we obtain at least one vector of $V_i$. For the three lines in the $i$-th triple of $\mathcal{T}$, let $\Pi$ be a plane given by Lemma 9. Let $\Pi'$ and $\Pi''$ be two planes parallel and arbitrarily close to $\Pi$, such that $\Pi$ is in the middle of them. Note that $v_{\Pi'}[i-1]$ equals $v_{\Pi''}[i-1]$, and that $v_{\Pi'}[i]$ differs from $v_{\Pi''}[i]$ their $i$-th coordinate. Thus we obtain two vectors from $v_{\Pi'}[i-1] = v_{\Pi''}[i-1]$. Therefore $|V_i| \geq |V_{i-1}| + 1$ and $|V_{\binom{n}{3}}| \geq \binom{n}{3}$. Since $V = V_{\binom{n}{3}}$ the result follows. □

**Theorem 11** *For any set $\mathcal{L}$ of $n$ lines, the total number of order types obtainable from all (general) cross-sections is at most $\mathcal{O}(n^9)$.*

**Proof.** Consider the following general parametrization of $\mathcal{L}$.

$$\ell_i \quad \equiv \quad (x, y, z) = (a_i, b_i, c_i) + \lambda_i(u_i, v_i, w_i), \quad 1 \leq i \leq n$$

Let, without loss of generality, $\ell_n$ be the vertical line through the origin. Further, consider a generic non-vertical plane

$$\Pi_{\alpha,\beta,\gamma} \equiv \alpha x + \beta y + z = \gamma$$

(note that vertical planes do not intersect $\ell_n$ in one point). Then for every $1 \leq i \leq n$, the intersection of $\Pi_{\alpha,\beta,\gamma}$ with $\ell_i$ is

$$\Pi_{\alpha,\beta,\gamma} \cap \ell_i \quad \equiv \quad \alpha(a_i + \lambda_i u_i) + \beta(b_i + \lambda_i v_i) + (c_i + \lambda_i w_i) = \gamma.$$

If $\Pi_{\alpha,\beta,\gamma}$ is not parallel to $\ell_i$, then $(\alpha u_i + \beta v_i + w_i) \neq 0$ and we obtain the following as an equivalent expression where $\lambda_i$ is explicit.

$$\Pi_{\alpha,\beta,\gamma} \cap \ell_i \quad \equiv \quad \lambda_i = \frac{\gamma - \alpha a_i - \beta b_i - c_i}{\alpha u_i + \beta v_i + w_i}$$

Assuming that $\Pi_{\alpha,\beta,\gamma}$ is not parallel to any line of $\mathcal{L}$ we can consider the cross-section $\{p_1, \ldots, p_n\}$ of $\mathcal{L}$ with $\Pi_{\alpha,\beta,\gamma}$. As $\Pi_{\alpha,\beta,\gamma}$ is not vertical, the orientation of three points $p_i$, $p_j$, and $p_k$ in the cross-section is the same as the orientation of the projection of $p_i$, $p_j$, $p_k$ to the horizontal plane $z = 0$. The latter is expressed by (the sign of) the determinant

$$\Delta_{i,j,k}(\alpha,\beta,\gamma) \quad = \quad \begin{vmatrix} a_i + \lambda_i u_i & b_i + \lambda_i v_i & 1 \\ a_j + \lambda_j u_j & b_j + \lambda_j v_j & 1 \\ a_k + \lambda_k u_k & b_k + \lambda_k v_k & 1 \end{vmatrix}.$$

The three points $p_i$, $p_j$, and $p_k$ are collinear in the cross-section if and only if $\Delta_{i,j,k}(\alpha,\beta,\gamma) = 0$. Further, we have

$$\Delta_{i,j,k}(\alpha,\beta,\gamma) = \frac{P_{i,j,k}(\alpha,\beta,\gamma)}{\prod_{x \in \{i,j,k\}} (\alpha u_x + \beta v_x + w_x)},$$

where $P_{i,j,k}(\alpha,\beta,\gamma)$ is a polynomial in the variables $\alpha$, $\beta$, and $\gamma$ of degree at most three. The denominator of this fraction is zero if and only if $\Pi_{\alpha,\beta,\gamma}$ is parallel to one of the lines $\ell_i$, $\ell_j$, and $\ell_k$ (and hence non-zero for all considered planes). Thus, $\Delta_{i,j,k}(\alpha,\beta,\gamma) = 0$ if and only if $P_{i,j,k}(\alpha,\beta,\gamma) = 0$.

Now consider the parametric space $\mathcal{S}$ in which every plane $\Pi_{\alpha,\beta,\gamma}$ is represented by the point $(\alpha,\beta,\gamma)$. In $\mathcal{S}$, $P_{i,j,k}(\alpha,\beta,\gamma)$ is a polynomial surface of maximum degree three. Further the set of non-vertical planes that are parallel to line $\ell_i$,

$$\delta_i(\alpha,\beta,\gamma) \quad \equiv \quad \alpha u_i + \beta v_i + w_i = 0,$$

is a surface of maximum degree one in $\mathcal{S}$.

We consider the arrangement $\mathcal{A}$ of the $\binom{n}{3}$ polynomial surfaces $P_{i,j,k}(\alpha,\beta,\gamma)$, with $1 \leq i < j < k \leq n$, and the $n-1$ polynomial surfaces $\delta_i(\alpha,\beta,\gamma)$, with $1 \leq i \leq n-1$, in $\mathcal{S}$. As $\mathcal{A}$ is an arrangement of $\mathcal{O}(n^3)$ objects in three-dimensional space, it has $\mathcal{O}(\binom{n^3}{3}) = \mathcal{O}(n^9)$ cells.

Further, for all planes $\Pi_{\alpha,\beta,\gamma}$ corresponding to (the interior of) a cell in $\mathcal{A}$, the order type of the cross-section of $\mathcal{L}$ with $\Pi_{\alpha,\beta,\gamma}$ is the same. Hence, the number of order types that can be obtained from cross-sections of $\mathcal{L}$ is $\mathcal{O}(n^9)$. □

The upper bound from the previous theorem is asymptotically tight, as the following theorem proves.

**Theorem 12** *There exist sets $\mathcal{L}$ of $n$ lines and directions $\vec{\ell}$ such that the number of different order types obtained from cross-sections is at least $\Omega(n^9)$.*

**Proof.** Consider again the sets $\mathcal{L}$ from the proof of Theorem 3. When sweeping $\mathcal{L}$ in the direction $\vec{\ell}$ that gives $\Omega(n^3)$ different order types, every order type is obtained for a certain interval with respect to the sweeping direction. Thus, each plane obtaining a certain order type can slightly be wiggled in without changing the obtained order type. Now let the cardinality of $\mathcal{L}$ be $\frac{n}{3}$ and consider a set $\mathcal{L}'$ which consists of three copies of $\mathcal{L}$ that are arranged essentially parallel and very far from each other (like "thick edges" of an equilateral prism); see Figure 3 for a sketch.

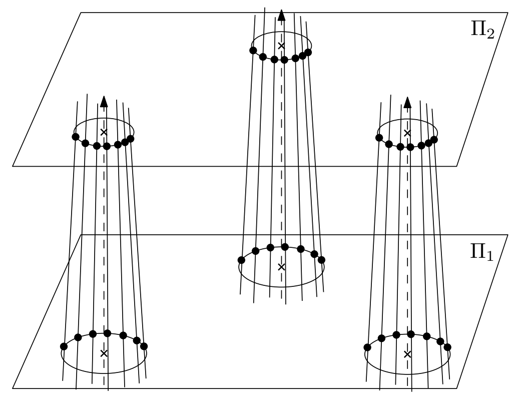


Figure 3: A set of lines and a direction where $\binom{n}{3}+1$ different order types are encountered with planes orthogonal to $\vec{\ell}$.

Then for any 3-combination of order types obtainable from one set $\mathcal{L}$ there is a plane obtaining them simultaneously, which implies a lower bound of $\binom{\frac{n}{3}}{3}^3$ for the number of order types obtained by this set. □

**Corollary 13** *The maximum number of different order types obtainable from a set of $n$ lines in general position is $\Theta(n^9)$.*

One fact that we have used repeatedly throughout this work is that, given an arrangement $\mathcal{L}$ of lines in general position and a plane $\Pi$ that induces a general cross-section of $\mathcal{L}$, it is always possible to slightly "wiggle" $\Pi$ without changing the order type of the cross-section. A change of the order type can occur only if, during the wiggling, either three points of the cross-section become collinear or $\Pi$ becomes parallel to a line of $\mathcal{L}$.

One question following from these observations is the following: Given an arrangement $\mathcal{L}$ of lines and plane $\Pi$ that induces a general cross-section of $\mathcal{L}$, how many different order types can be reached "directly" by wiggling $\Pi$? More exactly, consider a continuous motions of $\Pi$ during which any occurrence of a plane not inducing a general cross-section is a singular event. How many different order types can be obtained by such continuous motions that contain exactly one such singular event?

Consider the unique partition of the (infinite) set of planes that induce general cross-sections into maximal (infinite) subsets such that every subset is closed under continuous motion (i.e., every plane can be transformed to every other by continuous motion, without any occurrence of a non-general cross-section during this motion). Note that all elements of one set of this partition induce the same order type. To the contrary, different sets need not induce different order types.

Let the *mutation graph (of $\mathcal{L}$)* be the graph where every vertex corresponds to one set of this partition. Further, two vertices (sets of planes) are connected by an edge if and only if there is some continuous motion from a plane of the one set to a plane of the other set that contains exactly one non-general cross-section. The following proposition states that the mutation graph does never have too many edges.

**Proposition 14** *The mutation graph of any arrangement $\mathcal{L}$ of lines in general position has an average vertex degree of $\Theta(1)$.*

For an arrangement of lines $\mathcal{L}$, consider again a direction $\vec{\ell}$ and the sequence of order types that is obtained when sweeping $\mathcal{L}$ in the direction of $\vec{\ell}$. We say that two sweeping directions $\vec{\ell}$ and $\vec{\ell'}$ are *different with respect to $\mathcal{L}$*, if the two sequences of order types resulting from sweeping $\mathcal{L}$ in direction of $\vec{\ell}$ and $\vec{\ell'}$, respectively, are different.

**Theorem 15** *For any line arrangement $\mathcal{L}$ with cardinality $n$, the number of different sweeping directions with respect to $\mathcal{L}$ is $\mathcal{O}(n^{12})$.*

Due to space constraints, the proofs of Proposition 14 and Theorem 15 are deferred to the full version. For both proofs, an essential ingredient is the arrangement $\mathcal{A}$ of polynomial surfaces from the proof of Theorem 12.

## 4 Concluding remarks

In a full version of this work we also consider other combinatorial properties of points sets that arise as cross-sections of line arrangements in 3-dimensional space, and we also extend our results to plane sections of arrangements of $(d-2)$-flats in $\mathbb{R}^d$.

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