Optimizing some constructions with bars: new geometric knapsack problems

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Abstract

A set of vertical bars planted on given points of a horizontal line defines a *fence* composed of the quadrilaterals bounded by successive bars. A set of bars in the plane, each having one endpoint at the origin, defines an *umbrella* composed of the triangles bounded by successive bars. Given a collection of bars, we study how to use them to build the fence or the umbrella of maximum total area. We present optimal algorithms for these constructions. The problems introduced in this paper are related to the Geometric Knapsack problems [Arkin et al., Algorithmica 1993] and the Rearrangement Inequality [Wayne, Scripta Math. 1946].

Keywords: Geometric optimization; Combinatorial optimization; Inequalities; Optimal algorithms.

1 Introduction

In this paper we introduce the following geometric optimization problem. Given a collection $S = \{s_0, s_1, \ldots, s_{n-1}\}$ of n line segments or bars with lengths $\ell_0, \ell_1, \ldots, \ell_{n-1}$, respectively, and a set $P = \{x_0 < x_1 < \ldots < x_{n-1}\}$ of n points on the x-axis, locate the segments vertically on the given points such that each point receives a segment and the polygon with vertices the endpoints of the segments has maximum area. See Figure 1. Notice that this problem can be seen as an ordering problem that is built around a profit function f that assigns to every vertical segment s_i the corresponding real profit $f(s_i)$. The profit $f(s_i)$ is given by the area of the quadrilateral bounded by the segments s_{i-1} and s_i if $i \geq 1$, and 0 otherwise. The total profit of an ordering is

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the sum of the profits of the individual segments in S. We also consider the case where $P = \{x_0\}$ is composed by one point and the segments are not necessarily located vertically on x_0. See Figure 2 where all segments has a common endpoint, x_0. We study some variants of the problem that allow to construct with n bars of different lengths certain geometric object of maximum area. Formally, the problems are defined as follows.

1.1 Problems formulation

Let OX denote the x-axis. We say that a bar s is *planted* if s is vertical and the bottom endpoint of s belongs to OX. Every set of n planted bars $s_0, s_1, \ldots, s_{n-1}$ (from left to right) induces an x-monotone polygon of $n+2$ vertices consisting of the bottom endpoints of s_0 and s_{n-1}, and the top endpoints of the n bars. Such a polygon is called a *fence*. Given a set P of n points $x_0 < x_1 < \ldots < x_{n-1}$ on OX and a vector (or sequence) $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$ of n lengths, we denote by $F(P, L)$ the fence induced by planted bars $s_0, s_1, \ldots, s_{n-1}$, where s_i has length ℓ_i and bottom endpoint x_i, for $i = 0, 1, \ldots, n-1$. Refer to Figure 1.

Given a set P of n points $x_0 < x_1 < \ldots < x_{n-1}$ on OX and a sequence $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$ of n lengths, we study the next two problems considering fences:

The Maximum Fence problem: Find a permutation L' of L such that the fence $F(P, L')$ has maximum area.

The Maximum Convex Hull Fence problem: Find a permutation L' of L such that the convex hull of the fence $F(P, L')$ has maximum area.

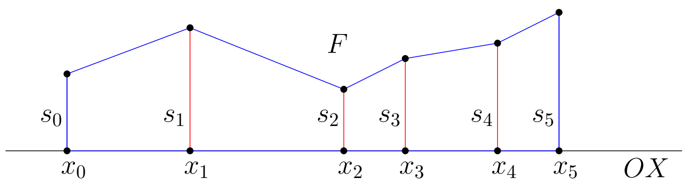

Figure 1: The fence F is induced by the bars $s_0, s_1, \ldots, s_5$ planted on $x_0, x_1, \ldots, x_5$, respectively. If ℓ_i is the length of s_i for every i then F is the fence $F(\{x_0, \ldots, x_5\}, \langle \ell_0, \ldots, \ell_5 \rangle)$.

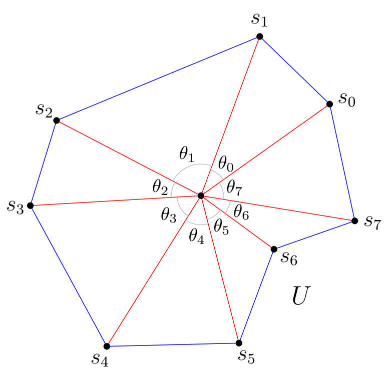

Figure 2: The umbrella $U := U(\Theta, L)$ for $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_7 \rangle$ and $L = \langle \ell_0, \ell_1, \ldots, \ell_7 \rangle$, where ℓ_i is the length of bar s_i for $i = 0, \ldots, 7$.

An *umbrella* is a star-shaped polygon having the origin of coordinates as center. A polygon Q is *star-shaped* if it contains a point p such that for all points q of Q the straight segment joining p and q is contained in Q. The point p is called a *center* of Q. Every set of n bars, each of them having an endpoint at the origin of coordinates and such that every angle between successive bars is less than π, induces an umbrella which has as vertices the endpoints of the bars distinct from the origin. When considering umbrellas we assume $n \geq 3$. Given an integer n, an *n-partition* is any sequence of n non-negative angles so that each of them is less than π and their sum is equal to 2π. Given an n-partition $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ and a sequence $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$ of n lengths, the umbrella $U(\Theta, L)$ is an umbrella induced by n bars $s_0, \ldots, s_{n-1}$ with an endpoint at the origin and lengths $\ell_0, \ldots, \ell_{n-1}$, respectively, such that $s_0, \ldots, s_{n-1}$ are radially sorted in counterclockwise order around the origin and the angle between bars s_i and s_{i+1} is equal to θ_i for $i = 0, 1, \ldots, n-2$. Refer to Figure 2.

We study the following problem considering umbrellas:

The Maximum Umbrella problem: Given an n-partition $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ and a sequence $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$ of n lengths, find both a permutation Θ' of Θ and a permutation L' of L such that the area of the umbrella $U(\Theta', L')$ is maximized.

1.2 Results

We show that the Maximum Fence problem can be solved in $\Theta(n \log n)$ time. We prove that this time complexity is optimal in the algebraic decision tree model by using a reduction from the Sorting problem which has lower bound $\Omega(n \log n)$ in this model [3]. The solution we propose is related to the Rearrangement Inequality [12]. For the Maximum Convex Hull Fence problem we present a linear time algorithm.

We show that optimal solutions of the Maximum Umbrella problem satisfy monotone properties on both the bar lengths and the sines of the angles between succesive bars. In order to do this, we first solve a problem (called the Two Permutations problem) considering the optimal permutation of two vectors, which is new to our knowledge. We use the Extended Rearrangement Inequality [1] to solve such a problem. The monotone properties of the Maximum Umbrella problem allow us to solve it in $\Theta(n \log n)$ time. We prove that this time complexity is optimal in the algebraic decision tree model by using again a reduction from the Sorting problem.

1.3 Related work: the geometric knapsack problems

Although the geometric optimization problems introduced in this paper are interesting in its own right, independent of any applications they might find, we show that the problems can be considered related to the geometric knapsack problems.

Knapsack problems have been extensively studied in applied mathematics [10]. These problems are in the class of combinatorial optimization problems and the name is derived from the maximization problem of best selection of essentials that can fit into a bag to be carried on a tour. One of its classical version is defined as follows. We are given n objects and a knapsack. Let w_i be the weight of object i and W be the capacity of the knapsack. If the i-th item is placed into de knapsack then a profit p_i is earned. The objective is to fill the knapsack so that the maximum profit is earned.

Arkin et al. [2] mapped classical knapsack problems into a new class of Geometric Knapsack problems. For their purposes, a knapsack is a simple closed curve whose capacity is its length or the area it encloses, and objects may be points, polygons, line segments, etc. The net profit is defined as the sum of the values of the item enclosed by the curve minus the cost (if any) of the curve used. Notice that in the classical Knapsack problem, the selection of an item depends upon its weight and capacity. But in Geometric Knapsack problems, the selection of an item depends not only upon its weight and capacity but also on the position of other items. This geometric variant leads to a new class of algorithmic problems that can be addressed with computational geometry tools. Moreover, with a similar mapping, it can be shown that a wide class of problems in geometric optimization and facility location can be viewed as geometric knapsack problems. We review here some examples.

Let us consider the Maximum Covering Location problem (MCLP), originally stated by Church and Velle [5]. A limited number of facilities are installed and the goal is to maximize the coverage (population covered) within a given covering distance by selecting a fixed number p of facilities. This problem can be viewed as a geometric multiple knapsack problem as follows. The profit is represented by the covered demand and the knapsack by a limited number of facilities to be installed. The planar maximum covering problems, where facilities can be placed anywhere on the plane, have also been considered. For instance, for Euclidean (rectilinear) distances, the candidate facility locations would be the points of intersection of circles (diamonds) centered at the demand points [6].

Another closely related problem is to locate one or more convex objects (squares, rectangles, convex polygon, circles, etc.) to maximize the size of the subset covered. Given a set of n points with arbitrary weight and one (or some) objects of fixed size, find the placement of the objects that maximizes the sum of the weights of the enclosed points. Some of these variants for different geometric objects can be found in the literature: convex polygon [7], disk and rectangle [9], unit disks and squares for bichromatic point sets [4]. We refer to the works of Plastria [11], and Drezner and Hamacher [8], for two comprehensive studies of covering location problems.

The problems addressed in this paper can also be viewed as a variant of the geometric multiple knapsack problems where the profit is represented by the area of the built region by means of the location of vertical segments. More specifically, a set of points are given on the x-axis and the goal is to built the fence with maximum area (profit) by locating a set of bars (knapsacks) onto the given points. As we will show in Section 5, the problems introduced in this paper lead to consider other variants that are of interest in the combinatorial and geometric optimization area.

1.4 Outline

The paper is outlined as follows. In Section 2 we present the Rearrangement Inequality and the Extended Rearrangement Inequality. In Section 3 the optimal running-time algorithms solving the Maximum Fence problem and the Maximum Convex Hull Fence problem are given. In Section 4 we study the Maximum Umbrella problem, and this section is divided into two subsections, Subsection 4.1 and Subsection 4.2. In the former one the Two Permutations problem is solved, and in the latter one the main result considering the MAXIMUM UMBRELLA problem is given. Finally, in Section 5, both the conclusions and several further research directions are stated.

2 Preliminaries

Given $2n$ real values $x_1 \leq \cdots \leq x_n, y_1 \leq \cdots \leq y_n$, the *Rearrangement Inequality* [12] states that

$$x_n y_1 + \cdots + x_1 y_n \leq x'_1 y_1 + \cdots + x'_n y_n \leq x_1 y_1 + \cdots + x_n y_n$$

for any permutation $\langle x'_1, x'_2, \ldots, x'_n \rangle$ of $\langle x_1, x_2, \ldots, x_n \rangle$.

The Rearrangement Inequality implies a straightforward solution of the following problem called MAXIMUM VECTOR PRODUCT problem: Given two vectors $a, b \in \mathbb{R}^n$, find both a permutation $a' = \langle a'_0, a'_1, \ldots, a'_{n-1} \rangle$ of a and a permutation $b' = \langle b'_0, b'_1, \ldots, b'_{n-1} \rangle$ of b such that the scalar product $a' \cdot b' = \sum_{i=0}^{n-1} a'_i \cdot b'_i$ is maximized.

Proposition 1 *The* MAXIMUM VECTOR PRODUCT *problem can be solved in $O(n \log n)$ time by sorting the components of a and the components of b, i.e. $a'_0 \leq a'_1 \leq \ldots \leq a'_{n-1}$ and $b'_0 \leq b'_1 \leq \ldots \leq b'_{n-1}$.*

Angell [1] proved a generalization of the Rearrangement Inequality called the Extended Rearrangement Inequality.

Theorem 2 (*Extended Rearrangement Inequality* [1]) *Let $m \geq 1$ be an integer number and $y_{i,1} \leq y_{i,2} \leq \ldots \leq y_{i,n}$ $(i = 1, 2, \ldots, m)$ and $z_1 \leq z_2 \leq \ldots \leq z_n$ be $m + 1$ sequences of n non-negative numbers. Then:*

$$\sum_{j=1}^{n} y'_{1,j} y'_{2,j} \ldots y'_{m,j} z_j \leq \sum_{j=1}^{n} y_{1,j} y_{2,j} \ldots y_{m,j} z_j$$

for any permutation $\langle y'_{i,1}, y'_{i,2}, \ldots, y'_{i,n} \rangle$ of $\langle y_{i,1}, y_{i,2}, \ldots, y_{i,n} \rangle$, $i = 1, 2, \ldots, m$.

We will apply the Extended Rearrangement Inequality several times for $m = 2$. The proof is provided for completeness.

Proposition 3 (*2-Extended Rearrangement Inequality*) *Let $x_1 \leq \cdots \leq x_n$, $y_1 \leq \cdots \leq y_n$, and $z_1 \leq \cdots \leq z_n$ be there sequences of non-negative numbers. Then:*

$$x'_1 y'_1 z_1 + \cdots + x'_n y'_n z_n \leq x_1 y_1 z_1 + \cdots + x_n y_n z_n \qquad (1)$$

for any permutation $\langle x'_1, x'_2, \ldots, x'_n \rangle$ of $\langle x_1, x_2, \ldots, x_n \rangle$ and any permutation $\langle y'_1, y'_2, \ldots, y'_n \rangle$ of $\langle y_1, y_2, \ldots, y_n \rangle$.

Proof. Let us use induction on n. For $n = 1$ the result is immediate. Consider $n > 1$. Then we have:

$$\begin{aligned} & x'_1y'_1z_1 + \cdots + x'_ny'_nz_n \\ = & (x'_1y'_1 + \cdots + x'_ny'_n)z_1 + \\ & x'_2y'_2(z_2 - z_1) + \ldots + x'_ny'_n(z_n - z_1) \end{aligned} \tag{2}$$

Observe that $0 \le z_2 - z_1 \le \ldots \le z_n - z_1$. Then we have from the inductive hypothesis, $0 \le x_1 \le \cdots \le x_n$, and $0 \le y_1 \le \cdots \le y_n$, that:

$$\begin{aligned} & x'_2y'_2(z_2 - z_1) + \ldots + x'_ny'_n(z_n - z_1) \\ \le & x''_2y''_2(z_2 - z_1) + \ldots + x''_ny''_n(z_n - z_1) \\ \le & x_2y_2(z_2 - z_1) + \ldots + x_ny_n(z_n - z_1) \end{aligned} \tag{3}$$

where $\langle x''_2, \ldots, x''_n\rangle$ and $\langle y''_2, \ldots, y''_n\rangle$ are permutations of $\langle x'_2, \ldots, x'_n\rangle$ and $\langle y'_2, \ldots, y'_n\rangle$, respectively, satisfying $x''_2 \le \ldots \le x''_n$ and $y''_2 \le \ldots \le y''_n$. We also have

$$(x'_1y'_1 + \cdots + x'_ny'_n)z_1 \le (x_1y_1 + \cdots + x_ny_n)z_1 \tag{4}$$

from the Rearrangement Inequality and $z_1 \ge 0$. Therefore, we obtain Equation (1) from Equations (3) and (4). □

3 Fence problems

In this section we present optimal time algorithms for the Maximum Fence problem and the Maximum Convex Hull Fence problem.

Theorem 4 *The* Maximum Fence *problem can be solved in $\Theta(n \log n)$ time, which is optimal in the algebraic desicion tree model.*

Proof. Let P be a set of n points $x_0 < x_1 < \ldots < x_{n-1}$ on OX and $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1}\rangle$ be a sequence of n lengths. Define $d_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n-1$ and let $L' = \langle \ell'_0, \ell'_1, \ldots, \ell'_{n-1}\rangle$ be any permutation of L. The area of the fence $F(P, L')$ is equal to:

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{n-1} d_i(\ell'_{i-1} + \ell'_i) &= \frac{1}{2}(\ell'_0d_1 + \ell'_1(d_1 + d_2) + \ldots + \\ & \ell'_{n-2}(d_{n-2} + d_{n-1}) + \ell'_{n-1}d_{n-1}) \end{aligned}$$

Therefore, the problem is equivalent to solving the Maximum Vector Product problem for input $\langle d_1, d_1 + d_2, d_2 + d_3, \ldots, d_{n-2} + d_{n-1}, d_{n-1}\rangle$ and $\langle \ell_0, \ell_1, \ldots, \ell_{n-1}\rangle$, which can be solved in $O(n \log n)$ time by Proposition 1. This algorithm is optimal because what follows is a linear-time reduction from the Sorting problem. Let the set $X = \{y_0, y_1, \ldots, y_{n-1}\}$ of n elements be an instance of the Sorting problem. We construct an instance of the Maximum Fence problem consisting of n points $x_0 < x_1 < \ldots < x_{n-1}$ of OX such that $x_0 = 0$ and $x_i = x_{i-1} + i$ for $i = 1, \ldots, n-1$ and $L = \langle y_0, \ldots, y_{n-1}\rangle$. Observe that $d_1 < d_1 + d_2 < d_2 + d_3 < \ldots < d_{n-2} + d_{n-1}$. Then an optimal permutation $L' = \langle y'_0, y'_1, \ldots, y'_{n-1}\rangle$ of L solving the Maximum Fence problem satisfies $y'_0 < y'_1 < \ldots < y'_{n-2}$. Thus a sorting of X can be obtained in linear time by inserting orderly the element y'_{n-1} in the sorted sequence $\langle y'_0, y'_1, \ldots, y'_{n-2} \rangle$. □

Theorem 5 *The* MAXIMUM CONVEX HULL FENCE *problem can be solved in $O(n)$ time.*

Proof. Let P be a set of n points $x_0 < x_1 < \ldots < x_{n-1}$ on OX and $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$ be a sequence of n lengths. In the following we prove that the first and last elements of any optimal permutation of L are the two largest lengths of L. Let h_0 be the largest and h_1 be the second largest elements of L.

Let $L' = \langle \ell'_0, \ell'_1, \ldots, \ell'_{n-1} \rangle$ be any permutation of L, and $0 = i_0 < i_1 < \ldots < i_{k-1} = n-1$ be the indices such that the fence $F(\{x_{i_0}, x_{i_1}, \ldots, x_{i_{k-1}}\}, \langle \ell'_{i_0}, \ell'_{i_1}, \ldots, \ell'_{i_{k-1}} \rangle)$ is equal to the convex hull of $F(P, L')$. Define $d'_j = x_{i_j} - x_{i_{j-1}}$ for $j = 1, 2, \ldots, k-1$. The area of the convex hull of $F(P, L')$ is equal to:

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{k-1} d'_j (\ell'_{i_{j-1}} + \ell'_{i_j}) &\leq \frac{1}{2}(h_0 + h_1) \sum_{j=1}^{k-1} d'_j \\ &= \frac{1}{2}(h_0 + h_1)(x_{n-1} - x_0) \end{aligned}$$

The last expression is the area of the convex hull of any fence whose two largest bars have x_0 and x_{n-1} as bottom endpoints, respectively. Then, by finding the two largest lengths of L, we can build in linear time an optimal solution of the MAXIMUM CONVEX HULL FENCE problem. The result thus follows. □

4 The MAXIMUM UMBRELLA problem

In this section the MAXIMUM UMBRELLA problem is solved. From this point forward, all subindices are taken modulo n. Given two indices i and j, $0 \leq i, j < n$, we denote by $[i, j]$ the set of indices $\{i, i+1, \ldots, j-1, j\}$.

The MAXIMUM UMBRELLA problem finds an umbrella with maximum area, built from input collections of bar lengths and angles. If $\Theta' = \langle \theta'_0, \theta'_1, \ldots, \theta'_{n-1} \rangle$ and $L' = \langle \ell'_0, \ell'_1, \ldots, \ell'_{n-1} \rangle$ are an optimal solution for the instance $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ and $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$, then the area of the umbrella $U(\Theta', L')$ is equal to

$$\frac{1}{2} \sum_{i=0}^{n-1} \ell'_i \cdot \ell'_{i+1} \cdot \sin \theta'_i \tag{5}$$

Given an n-partition $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$, we denote by Θ_{sin} the sequence obtained by replacing each element θ_i of Θ by $\sin \theta_i$, that is, $\Theta_{sin} = \langle \sin \theta_0, \sin \theta_1, \ldots, \sin \theta_{n-1} \rangle$.

Observe that given (Θ, L) as input, the MAXIMUM UMBRELLA problem consists in rearranging the sequences Θ_{sin} and L into the sequences Θ'_{sin} and L', respectively, so that the expression (5) is maximized. A natural generalization of this rearrangement problem, called the TWO PERMUTATIONS problem, is studied in the next section. The algorithm to solve the TWO PERMUTATIONS problem will be used to solve the MAXIMUM UMBRELLA problem.

4.1 The Two Permutations problem

Given two sequences $a = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ and $b = \langle b_0, b_1, \ldots, b_{n-1} \rangle$ of non-negative numbers each, the Two Permutations problem consists in finding both a permutation $a' = \langle a'_0, a'_1, \ldots, a'_{n-1} \rangle$ of a and a permutation $b' = \langle b'_0, b'_1, \ldots, b'_{n-1} \rangle$ of b so that to maximize the expression

$$P(a', b') := \sum_{i=0}^{n-1} a'_i a'_{i+1} b'_i$$

Observe that if $a^* = \langle a'_0, a'_1, \ldots, a'_{n-1} \rangle$ and $b^* = \langle b'_0, b'_1, \ldots, b'_{n-1} \rangle$ form an optimal solution of the Two Permutations problem, then for every k, $1 \le k < n$, $a' = \langle a'_k, a'_{k+1}, \ldots, a'_{n-1}, a'_0, \ldots, a'_{k-1} \rangle$ and $b' = \langle b'_k, b'_{k+1}, \ldots, b'_{n-1}, b'_0, \ldots, b'_{k-1} \rangle$ are also an optimal solution. Furthermore, we have that $a'' = \langle a'_{n-1}, a'_{n-2}, \ldots, a'_0 \rangle$ and $b'' = \langle b'_{n-1}, b'_{n-2}, \ldots, b'_0 \rangle$ are an optimal solution. In other words, two optimal solutions of the Two Permutations problem are the same if one of them can be obtained by performing a rotation and/or inversion on the other one.

A *bitonic sequence* is a sequence $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ such that

$$x_0 \le x_1 \le \cdots \le x_k \ge x_{k+1} \ge \cdots \ge x_{n-1} \tag{6}$$

for some $k \in [0, n-1]$, or a rotation of such a sequence.

The Two Permutations problem is different from problems such as the Maximum Vector Product problem where the Rearrangement Inequality can be used directly. For example, for input $a = \langle 1, 2, 3, 4 \rangle$ and $b = \langle 1, 1, 1, 1 \rangle$ in which both sequences are sorted, we have $P(a, b) = 24$. However, for $a' = \langle 1, 3, 4, 2 \rangle$ and $b' = \langle 1, 1, 1, 1 \rangle$ we have $P(a', b') = 25 > P(a, b)$. We prove then the next lemma, stating that both sequences of an optimal solution must be bitonic.

Lemma 6 *Every optimal solution* (a^*, b^*) *of the* Two Permutations *problem is such that both* a^* *and* b^* *are bitonic.*

Proof. Let $a = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ and $b = \langle b_0, b_1, \ldots, b_{n-1} \rangle$ be an input of the Two Permutations problem. Consider an optimal solution (a^*, b^*) for this input. W.l.o.g. we assume $a^* = a$ and $b^* = b$. First, we show that a is bitonic.

Let a_i and a_j be a largest element and a smallest element of a, respectively. If $a_i \ge a_{i+1} \ge \ldots \ge a_j$ and $a_i \ge a_{i-1} \ge \ldots \ge a_j$ then a is bitonic. Suppose to the contrary that a is not bitonic. By symmetry we can assume that $a_i \ge a_{i+1} \ge \ldots \ge a_j$ does not hold. Then there exists an element a_k, $k \in [i+2, j-1]$, such that $a_{k-1} < a_k$. Notice that $a_k > a_j$. There can be many elements of $\{a_{i+2}, a_{i+3}, \ldots, a_{j-1}\}$ equal to a_k. Take then the last one a_t of such elements, thus we have $a_t > a_{t+1}$. Let a_{s+1} be the first element of $\{a_{i+1}, a_{i+2}, \ldots, a_{t-1}\}$ that is less than a_t. Then $a_s \ge a_t > a_{s+1}, a_{t+1}$, see Figure 3 (a).

We will make new sequences a' and b' by reversing the subsequence $\langle a_{s+1}, \ldots, a_t \rangle$ of a and a subsequence of b. Consider two cases by comparing the elements b_s and b_t. If $b_s \ge b_t$ then b' is

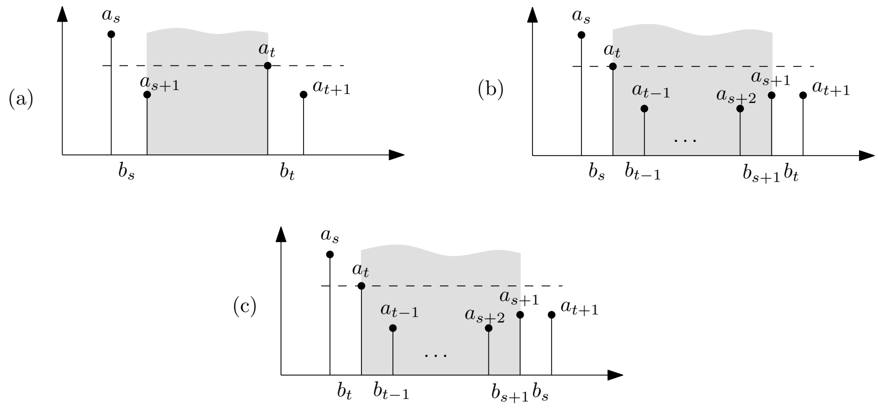

Figure 3: (a) A representation of the non-bitonic sequence a. The elements $a_0, \ldots, a_{n-1}$ of a are from left to right vertical bars of lengths $a_0, \ldots, a_{n-1}$, respectively. The space in between the bars of succesive elemens a_l and a_{l+1} is labeled with b_l. (b) New sequences a' and b' when $b_s \geq b_t$. (c) New sequences a' and b' when $b_s < b_t$.

obtained from b by reversing the subsequence $\langle b_{s+1}, b_{s+2}, \ldots, b_{t-1} \rangle$, see Figure 3 (b). Then

$$\begin{aligned} & P(a', b') - P(a, b) \\ = \; & a_s a_t b_s + a_{s+1} a_{t+1} b_t - a_s a_{s+1} b_s - a_t a_{t+1} b_t \\ = \; & (a_t - a_{s+1})(a_s b_s - a_{t+1} b_t) \\ > \; & 0 \end{aligned}$$

since $a_t > a_{s+1}$, $a_s \geq a_t > a_{t+1}$ and $b_s \geq b_t > 0$. This contradicts the assumption that pair a, b maximizes $P(a, b)$.

Case 2. $b_s < b_t$. We reverse the subsequences $\langle a_{s+1}, \ldots, a_t \rangle$ and $\langle b_s, \ldots, b_t \rangle$, see Figure 3 (c). Let a' and b' be the new sequences. Then

$$\begin{aligned} & P(a', b') - P(a, b) \\ = \; & a_s a_t b_t + a_{s+1} a_{t+1} b_s - a_s a_{s+1} b_s - a_t a_{t+1} b_t \\ = \; & (a_t b_t - a_{s+1} b_s)(a_s - a_{t+1}) \\ > \; & 0 \end{aligned}$$

since $a_t > a_{s+1}, b_t > b_s$ and $a_s \geq a_t > a_{t+1}$. This contradicts the assumption that the pair a, b maximizes $P(a, b)$.

Second, we show that b is bitonic. Again, let a_i and a_j be a largest element and a smallest element of a, respectively. Observe that $P(a, b)$ is equal to

$$P_{i,j}(a, b) + P_{j,i}(a, b)$$

where

$$\begin{aligned} P_{i,j}(a, b) = & a_i a_{i+1} b_i + a_{i+1} a_{i+2} b_{i+1} + \ldots + a_{j-1} a_j b_{j-1}, \\ P_{j,i}(a, b) = & a_j a_{j+1} b_j + a_{j+1} a_{j+2} b_{j+1} + \ldots + a_{i-1} a_i b_{i-1}. \end{aligned}$$

Since $P(a,b)$ is maximum and $a_i a_{i+1} \geq a_{i+1}a_{i+2} \geq \ldots \geq a_{j-1}a_j$ we conclude by the Rearrangement Inequality that

$$b_i \geq b_{i+1} \geq \ldots \geq b_{j-1}. \quad (7)$$

Similarly, since $P(a,b)$ is maximum and $a_j a_{j+1} \leq a_{j+1}a_{j+2} \leq \ldots \leq a_{i-1}a_i$, we conclude by the Rearrangement Inequality that

$$b_j \leq b_{j+1} \leq \ldots \leq b_{i-1}. \quad (8)$$

Equations (7) and (8) imply that the sequence b is bitonic. The lemma follows. □

Let (a^*, b^*) be an optimal solution of the Two Permutations problem. Lemma 6 states that the sequence a^* is bitonic. There are exponentially many bitonic sequences of size n. Next, we show that there are bitonic sequences a^* satisfying some properties up to rotations and inversions. We say that the solution (a^*, b^*) is *alternating* if $a^* = \langle \ldots, a_6, a_4, a_2, a_0, a_1, a_3, a_5, \ldots \rangle$ and $b^* = \langle \ldots, b_5, b_3, b_1, b_0, b_2, b_4, b_6, \ldots \rangle$, where $a_0 \geq a_1 \geq \ldots \geq a_{n-1}$, $b_0 \geq b_1 \geq \ldots \geq b_{n-1}$, and a_0 and b_0 are at the same position in a^* and b^*, respectively. Note that if (a^*, b^*) is alternating then both a^* and b^* are bitonic.

Lemma 7 *There always exists an alternating optimal solution of the* Two Permutations *problem.*

Proof. Consider an input of the the Two Permutations problem, consisting of a permutation of the elements $a_0 \geq a_1 \geq \ldots \geq a_{n-1}$ and a permutation of the elements $b_0 \geq b_1 \geq \ldots \geq b_{n-1}$.

We assume w.l.o.g. that a_0 is at position 0 of any bitonic permutation of a, thus a_1 must be at position either 1 or -1. Assume w.l.o.g. the former case. Let a^* denote the sequence $\langle \ldots, a_4, a_2, a_0, a_1, a_3, a_5, \ldots \rangle$ and b^* denote the sequence $\langle \ldots, b_5, b_3, b_1, b_0, b_2, b_4, \ldots \rangle$, where a_0 and b_0 are at the same position in a^* and b^*, respectively. Sequence a^* holds what we call the *alternating condition*: for $i = 2, \ldots, n-1$, a_i is at *alternating position* $-\lceil \frac{i}{2} \rceil$ if i is even or at *alternating position* $\lceil \frac{i}{2} \rceil$ if i is odd. Observe that every alternating permutation of a is equal to a^*.

We show by induction on $j \in [0, n-1]$ that, for every j, there is an optimal solution of the Two Permutations problem such that $a_0, a_1, \ldots, a_j$ are in alternating positions. By Lemma 6 this holds for $j = 0$. To prove the induction step, we consider an optimal solution (a,b) of the Two Permutations problem such that $a_0, a_1, \ldots, a_{j-1}$ are in alternating positions. We assume that j is odd since the case of even j can be treated similarly. If a_j is at the alternating position then we are done. Suppose that another element $a_k < a_j$, $k > j$, is at the alternating position of a_j. Since a is bitonic by Lemma 6, it must have the form

$$\langle \ldots, a_j, a_{j-1}, \ldots, a_4, a_2, a_0, a_1, a_3 \ldots, a_{j-2}, a_k, \ldots \rangle.$$

Let b_t be the element of b that is at the same position as a_j in a, and b_r be the element of b that is at the same position as a_{j-2} in a. Let $\langle \beta_1, \beta_2 \rangle$ be a permutation of $\langle b_t, b_r \rangle$ such that $\beta_1 \geq \beta_2$. Let a' be the sequence

$$\langle \ldots, a_k, a_{j-1}, \ldots, a_4, a_2, a_0, a_1, a_3 \ldots, a_{j-2}, a_j, \ldots \rangle$$

which is obtained from a by first reversing the subsequence $\langle a_{j-1}, a_{j-3}, \ldots, a_{j-4}, a_{j-2} \rangle$ and then reversing the whole a. If $b_t \geq b_r$ then let b' denote the sequence b. Otherwise, let b' denote the sequence obtained by swapping the elements b_t and b_r in b. Note then that $P(a', b') - P(a, b)$ is equal to

$$\begin{aligned} & a_k a_{j-1}\beta_2 + a_{j-2}a_j\beta_1 - (a_j a_{j-1} b_t + a_{j-2} a_k b_r) \\ = \;& a_k a_{j-1}\beta_2 + a_j a_{j-2}\beta_1 - (a_j a_{j-1} b_t + a_k a_{j-2} b_r) \end{aligned}$$

Since $a_k < a_j$, $a_{j-1} \leq a_{j-2}$, and $\beta_2 \leq \beta_1$, it follows from the definitions of β_1, β_2 and the 2-Extended Rearrangement Inequality that the last expression is greater than or equal to zero. Then $P(a', b') \geq P(a, b)$, implying that (a', b') is an optimal solution. When $j = n - 1$, then a' is alternating and $a' = a^*$.

We now proceed to prove that (a^*, b^*) is an optimal solution.

Let (a^*, b') be an optimal solution, where $b' = \langle \ldots, b'_5, b'_3, b'_1, b'_0, b'_2, b'_4, b'_6, \ldots \rangle$ and the position of b'_0 is the same as a_0 in a^*. Observe that

$$P(a^*, b') = a_0 a_1 b'_0 + a_0 a_2 b'_1 + a_1 a_3 b'_2 + a_2 a_4 b'_3 + a_3 a_5 b'_4 + \ldots$$

Since $a_0 a_1 \geq a_0 a_2 \geq a_1 a_3 \geq a_2 a_4 \geq a_3 a_5 \geq \ldots$ and $P(a^*, b')$ is maximum for all permutations b' of b, we must have $b'_0 \geq b'_1 \geq \ldots \geq b'_{n-1}$ by the Rearrangement Inequitably. Hence, (a^*, b^*) is an optimal solution and the result follows. □

We make the observation that Lemma 7 cannot be strengthen to state that every optimal solution is alternating. Indeed, consider the instance $a = \langle 6, 5, 5, 5, 5, 5, 4 \rangle$ and $b = \langle 1, 1, 1, 1, 1, 1, 1 \rangle$. Then $a^* = \langle 5, 5, 5, 6, 5, 4, 5 \rangle$ and $b^* = b$ form a non-alternating optimal solution. It is not hard to see, by using arguments similar to those used in Lemma 7's proof, that if all elements of input a are different, then every optimal solution is alternating.

Theorem 8 *The* Two Permutations *problem can be solved in $\Theta(n \log n)$ time, which is optimal in the algebraic desicion tree model.*

Proof. We can sort both a and b in $O(n \log n)$ time and obtain the solution described in Lemma 7. The algorithm is optimal because there is a linear time reduction from the sorting problem to the Two Permutations problem. The reduction is as follows. For every instance of the Sorting problem, consisting of a set $X = \{x_0, x_1, \ldots, x_{n-1}\}$ of n elements, we build the instance $a = \langle x_0, x_1, \ldots, x_{n-1} \rangle$ and $b = \langle 1, 1, \ldots, 1 \rangle$ of the Two Permutations problem. By Lemma 6, in the optimal solution (a^*, b^*) to the Two Permutations problem for input (a, b), a^* satisfies the bitonic property of Equation (6). The sorting of X can then be obtained in $O(n)$ time from a^*. □

4.2 Back to the Maximum Umbrella problem

Let an instance of the Maximum Umbrella problem consist of the n-partition $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ and the sequence $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$. Observe from Equation (5) that solving the Maximum Umbrella problem for this instance is equivalent to solving the Two Permutation problem for the instance $a = L$ and $b = \Theta_{sin}$. Let (Θ', L') be an optimal solution of the Maximum Umbrella problem for the instance (Θ, L). It also follows from Lemma 6 that the sequences Θ'_{sin} and L' are bitonic. Furthermore, from Lemma 7, it holds that Θ' and L' can have the form $\Theta' = \langle \ldots, \theta_3, \theta_1, \theta_0, \theta_2, \theta_4, \ldots \rangle$ and $L' = \langle \ldots, \ell'_4, \ell'_2, \ell'_0, \ell'_1, \ell'_3, \ldots \rangle$, respectively, where $\sin\theta'_0 \geq \sin\theta'_1 \geq \ldots \geq \sin\theta'_{n-1}$, $\ell'_0 \geq \ell'_1 \geq \ldots \geq \ell'_{n-1}$, and θ'_0 and ℓ'_0 are at the same position in Θ' and L', respectively. Hence we arrive to the next results:

Lemma 9 *For input* $\Theta = \langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ *and* $L = \langle \ell_0, \ell_1, \ldots, \ell_{n-1} \rangle$*, where* $\sin\theta_0 \geq \sin\theta_1 \geq \ldots \geq \sin\theta_{n-1}$ *and* $\ell_0 \geq \ell_1 \geq \ldots \geq \ell_{n-1}$*, an optimal solution of the* Maximum Umbrella *problem consists of the sequences* $\Theta' = \langle \ldots, \theta_3, \theta_1, \theta_0, \theta_2, \theta_4, \ldots \rangle$ *and* $L' = \langle \ldots, \ell_4, \ell_2, \ell_0, \ell_1, \ell_3, \ldots \rangle$*, where* θ_0 *and* ℓ_0 *are at the same position in* Θ' *and* L'*, respectively.*

Theorem 10 *The* Maximum Umbrella *problem can be solved in* $\Theta(n \log n)$ *time, which is optimal in the algebraic desicion tree model.*

5 Conclusions and further research

In this paper, we have introduced and solved new geometric optimization problems related to the Geometric Knapsack problems [2]. Some variants and extensions of the problems studied are of interest for further research. On one hand, some open problems related to umbrellas include:

- With the same input (Θ, L) of the Maximum Umbrella problem, find both a permutation Θ' of Θ and a permutation L' of L so that to maximize the area of the convex hull of the umbrella $U(\Theta', L')$.
- Given (Θ, L), find a permutation L' of L so that the area of the umbrella $U(\Theta, L')$ is maximized. In other words, if angles and lengths are given, and angles must have a fixed order, find a permutation of the lengths that induces the umbrella of maximum area.
- Given L, find both a n-partition Θ and a permutation L' of L so that the umbrella $U(\Theta, L')$ has maximum area. Here we have the freedom of selecting both the permutation of lengths and the permutation of angles.
- Given L, find a n-partition Θ so that umbrella $U(\Theta, L)$ has maximum area. Note that this variant is a restricted version of the previous one.

On the other hand, the fence problems can be extended to three dimensions as follows: We define a *tent* as a polyhedron with the following properties: One of its faces called *base* is a convex polygon contained in the xy-plane; the faces adjacent to the base are vertical trapezoids with at least two right angles; is z-monotone (i.e. any line vertical to the xy-plane intersects the interior of at most two faces of the tent); and the projection of its skeleton on the xy-plane is a triangulation of its base. Refer to Figure 4.

Let T be a tent. We call *node* every vertex of T not in the base of T. The orthogonal projections of the nodes of T on its base is called the *ground points* of T. The vertical segments joining the nodes of T with their corresponding ground points are called the *bars* of T. Let $G = \{p_0, p_1, \ldots, p_{n-1}\}$ be a set of n ground points in the xy-plane and L be a sequence of n bar lengths. For any permutation $L' = \langle \ell'_0, \ell'_2 \ldots, \ell'_{n-1} \rangle$ of L, we denote by $T(G, L')$ the tent generated by planting a bar of length ℓ'_i on the ground point p_i for $i = 0, 1, \ldots, n-1$. The optimization problems related to tents that we leave for further research are the following ones:

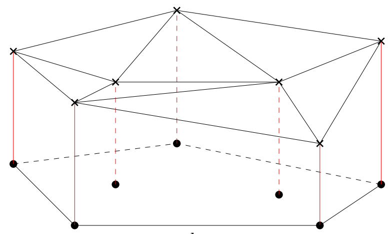

Figure 4: Example of a tent with seven ground points and seven bars. Ground points are denoted by tiny disks and nodes are denotes by crosses.

- Given a set G of n ground points and a sequence L of n bar lengths, find a permutation L' of L so that the volume of the tent $T(G, L')$ is maximized.
- Given a set G of n ground points and a sequence L of n bar lengths, find a permutation L' of L so that the volume of the convex hull of the tent $T(G, L')$ is maximized.

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